MST204.29

The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

Unit 29
Angular momentum and rigid bodies

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Angular momentum and rigid bodies

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First published 1982. New edition 1990. Reprinted 1993, 1995

Designed by the Graphic Design Group of the Open University.

Typeset by the TEXproject, Faculty of Mathematics, The Open University.

Printed and bound in the United Kingdom by Staples Printers Rochester Limited,
Neptune Close, Medway City Estate, Frindsbury, Rochester, Kent ME2 4LT

ISBN 0 7492 2035 X

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2.3

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Introduction

Why does a skater rotate more slowly when she stretches out her arms? Why does a spinning top balance on its point, when it would never do so if not spinning? The answers to these questions rely on the concept of *angular momentum*, which is the subject of this unit.

The same idea underlies an explanation of the experimental observations which you saw in the television programme for *Unit 27*. There two cylinders of the same mass and external dimensions rolled down a slope at different speeds. A similar outcome was observed for two spheres. In each case one of the bodies was solid and the other hollow, so that they had different *moments of inertia*. The theory to be developed here makes precise the effect of moment of inertia upon rotational motion. This effect is embodied in the *torque law*, which is the counterpart for rotational motion of Newton's second law for linear motion. The roles of mass, force and linear momentum are assumed in the rotational case by moment of inertia, torque and angular momentum respectively.

Unit 27 Subsection 2.1

The concept of *torque* was introduced in *Unit 28* Subsection 4.1.

All the results in this unit follow from Newton's laws of motion. As you will see, even the somewhat non-intuitive and surprising behaviour of a gyroscope is predictable as a consequence of these laws. This prediction was one of the triumphs in the nineteenth century application of Newtonian mechanics.

Study guide

Section 1 starts with a review of Newton's three laws of mechanics, culminating in the statement of a version of Newton's third law which is stronger (that is, more restrictive) than that introduced previously. Following the definition of *angular momentum*, the *torque law* is derived for a single particle. Then the corresponding law for a system of particles is obtained, by application of the strong form of Newton's third law.

A *rigid body* is a special type of particle system whose motion is considered in Section 2. After a reminder of how *moment of inertia* is defined, the torque law and kinetic energy for a rigid body are expressed in terms of its moment of inertia, and practice is provided in calculating this quantity. Often such calculation is assisted by use of a result known as the *parallel axes theorem*. In the absence of external torques, the angular momentum of a rigid body is conserved. After some applications of angular momentum conservation, the audio-tape subsection examines examples in which external torques are present.

Section 3 concentrates on rolling motions such as those of the *Unit 27* television programme, in which the axis of rigid-body rotation is no longer fixed but maintains a constant orientation. Both energy conservation and the torque law may be applied in this type of situation, whose analysis is aided by the *centre of mass decomposition theorems*.

The television programme in Section 4 investigates the motions of tops and gyroscopes, which are characterized by a changing orientation in the axis of rotation. The section concludes with further consideration of how a variable spin axis moves or *precesses*. Section 5 contains additional exercises for practice or revision purposes.

1 Angular momentum

1.1 Newton's laws revisited

The main purpose of this subsection is to return to a discussion of Newton's third law. This law played an important part in deriving a description for the linear motion of a many-particle system. Its role is equally crucial in obtaining the basic law for the angular motion of a many-particle system, as you will see shortly, but a slightly stronger version of the third law is now required.

Unit 17 Subsection 2.3

This review provides an opportunity for revision of all three of Newton's laws of motion, which are the cornerstone of classical mechanics. No result in any of the mechanics units of this course depends upon any theory beyond that implied by these three laws. Specific forces may be modelled in various ways. Thus the spring force is modelled as having magnitude proportional to the spring's extension (Hooke's law, *Unit 7*), while the force of gravitation exerted by one body upon another has magnitude inversely proportional to the square of the distance between the two bodies (Newton's universal law of gravitation, *Unit 30*). However, once these forces and the initial conditions are given, the complete behaviour of the system is determined simply by the application of Newton's laws of motion.

Newton's laws of motion were first described in the Introduction to *Unit 4*.

Law I states that, in the absence of forces, a particle either remains at rest or continues in uniform motion in a straight line. A reference system for which this law holds is called an *inertial frame*. For example, a reference frame fixed in an accelerating train would *not* be inertial, since an object placed 'at rest' on a smooth table in such a train would not remain at rest with respect to the frame. Any reference frame whose points move at uniform velocity with respect to an inertial frame is also an inertial frame (see Exercise 1(i) below).

In terms of the discussion of Newton's first law in *Unit 4* Subsection 2.4, an inertial frame is a coordinate system in which each point has the same constant velocity relative to the fixed stars.

Law II tells us that the acceleration experienced by a particle of constant mass is proportional to the force acting upon it, that is,

$$\mathbf{F} = m\mathbf{a},$$

where the force vector $\mathbf{F}$ and the acceleration vector $\mathbf{a}$ are both described by reference to an inertial frame set up in accordance with Law I. The constant of proportionality m is known as the *inertial mass* or simply the *mass* of the particle. If we introduce the linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$ of the particle, where $\mathbf{r}$ is its position vector, then this law may be written as

$$\mathbf{F} = \dot{\mathbf{p}}.$$

This law, arguably the most important of classical physics, enables us to deduce the motion of a particle from the force acting on it, by solution of the differential equation

$$m\ddot{\mathbf{r}} = \mathbf{F}.$$

Law II is valid in all frames which move with uniform velocity with respect to a given inertial frame in which it is valid (see Exercise 1(ii)).

Law III describes the mutual interaction between two particles, and so is indispensable for generalizing our mechanical laws from a single particle to a system of many, possibly interacting, particles. It tells us that the force $\mathbf{F}_{12}$ exerted on Particle 1 by Particle 2 is equal in magnitude but opposite in direction to the force $\mathbf{F}_{21}$ exerted on Particle 2 by Particle 1 (see Figure 1), that is,

$$\mathbf{F}_{12} = -\mathbf{F}_{21}.$$

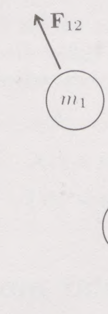
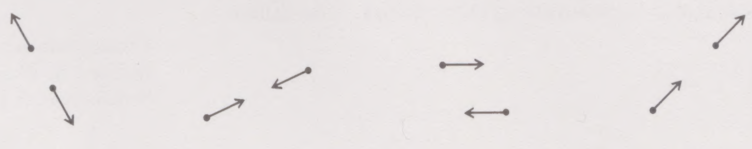


Figure 1

In fact, for the purposes of this unit we require a stronger form of the law, namely, that the forces are not only equal and opposite but also *act along the line joining the two particles*. This 'strong form' of Newton's third law is satisfied by the pair of inter-particle forces shown in Figure 1. Of the pairs of forces shown below, those in Figures 2(a) and 2(b) obey the strong form of the law, while that in Figure 2(c) satisfies only the weak form and that in Figure 2(d) does not satisfy even the weak form.



(a) (b) (c) (d)

Figure 2

> **Newton's third law (strong form)**
>
> For a pair of interacting particles, the force $\mathbf{F}_{12}$ on Particle 1 due to Particle 2 is equal in magnitude but opposite in direction to the force $\mathbf{F}_{21}$ on Particle 2 due to Particle 1, that is,
>
> $$\mathbf{F}_{12} = -\mathbf{F}_{21}.$$
>
> Each of these forces acts along the line which joins the two particles.

You will see later that it is necessary to assume the strong form of Newton's third law in order to derive the law of conservation of angular momentum for a system of (interacting) particles. Henceforth in this unit, any reference to Newton's third law is to be taken as meaning the *strong form* of the law.

Exercise 1

(i) Show that a reference frame which moves with uniform velocity $\mathbf{u}$ with respect to a given inertial frame is also inertial. In other words, supposing that a particle is at rest or in uniform motion in reference frame $Oxyz$, show that the particle is also at rest or in uniform motion with respect to reference frame $O'x'y'z'$, each point of which moves with uniform velocity $\mathbf{u}$ with respect to $Oxyz$ (see Figure 3).

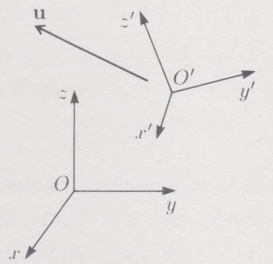


Figure 3

(ii) If Newton's second law

$$\mathbf{F} = m\ddot{\mathbf{r}} = m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k})$$

holds for a particle in $Oxyz$, show that it holds in the form

$$\mathbf{F} = m\ddot{\mathbf{r}}' = m(\ddot{x}'\mathbf{i}' + \ddot{y}'\mathbf{j}' + \ddot{z}'\mathbf{k}')$$

in $O'x'y'z'$, where $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of the particle with respect to $Oxyz$ and $O'x'y'z'$ respectively, and $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ are the Cartesian unit vectors in the second frame.

(iii) The frame $\{\mathbf{i}, \mathbf{j}\}$ with origin O is a (two-dimensional) inertial frame. The frame $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ has an origin which moves anticlockwise at a constant rate around a circle with centre at O, as shown in Figure 4. Here $\mathbf{e}_r, \mathbf{e}_\theta$ are unit vectors in the radial and transverse directions, as introduced in *Unit 28* Subsection 4.2. Decide whether the frame $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ is inertial.

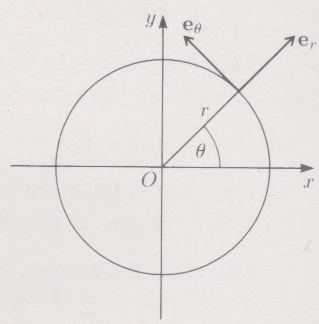


Figure 4

Exercise 2

Suppose that $\mathbf{F}_{ij}$ is the force exerted on Particle i due to Particle j, and that $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$ is the displacement vector from Particle i to Particle j $(i, j = 1, 2)$. In each of the following cases, decide whether either the strong or the weak form of Newton's third law is satisfied by the inter-particle forces, where λ and μ are constants.

(i) $\mathbf{F}_{12} = \lambda\mathbf{r}_{12}, \quad \mathbf{F}_{21} = \lambda\mathbf{r}_{21}.$

(ii) $\mathbf{F}_{12} = \mu\mathbf{r}_{12} \times \dot{\mathbf{r}}_{12}, \quad \mathbf{F}_{21} = \mu\mathbf{r}_{21} \times \dot{\mathbf{r}}_{21}.$

[*Solutions on page 48*]

1.2 Angular momentum and torque for a single particle

In order to introduce the concepts needed to analyse the motion of rotating objects, it is helpful to begin with the simple case of a single particle of mass m moving along a curve in space. This is illustrated in Figure 5. At time t the particle has position vector $\mathbf{r}(t)$ and velocity $\dot{\mathbf{r}}(t)$ relative to a chosen static coordinate system $Oxyz$. The linear momentum of the particle is

$$\mathbf{p}(t) = m\dot{\mathbf{r}}(t).$$

Linear momentum was defined in *Unit 17* Subsection 3.1.

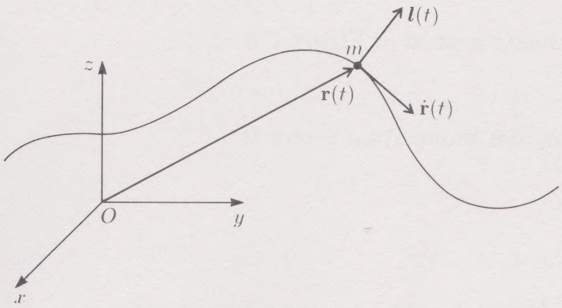


Figure 5

Without any attempt at prior intuitive justification, we shall now make a definition. At any given time t, the **angular momentum** $\boldsymbol{l}(t)$ of the particle, relative to the given origin O, is defined as the cross product

$$\boldsymbol{l}(t) = \mathbf{r}(t) \times \mathbf{p}(t) = \mathbf{r}(t) \times m\dot{\mathbf{r}}(t). \tag{1}$$

The SI units for the magnitude of angular momentum are $\mathrm{kg\,m^2\,s^{-1}}$.

There are two immediate comments to be made about this definition.

(i) We do not claim that it is an obvious development at this stage. You will see shortly why the vector $\boldsymbol{l}$ is of particular interest, but for the moment we shall just concentrate on the properties which follow from the definition given in Equation (1).

(ii) The position vector $\mathbf{r}$ which appears in the definition is dependent on the choice of origin for the coordinate system. Hence this is also true of the angular momentum vector. In Figure 6 below, the vector $\boldsymbol{l} = \mathbf{r} \times m\dot{\mathbf{r}}$ represents the angular momentum relative to the origin O, while the vector $\boldsymbol{l}' = \mathbf{r}' \times m\dot{\mathbf{r}}'$ represents the angular momentum relative to the origin O' of another static coordinate system. These vectors may differ in both magnitude and direction.

Since both coordinate systems are static, the velocity vector of the particle is the same in each case, as can be verified by putting $\mathbf{u} = \mathbf{0}$ into the solution of Exercise 1(i).

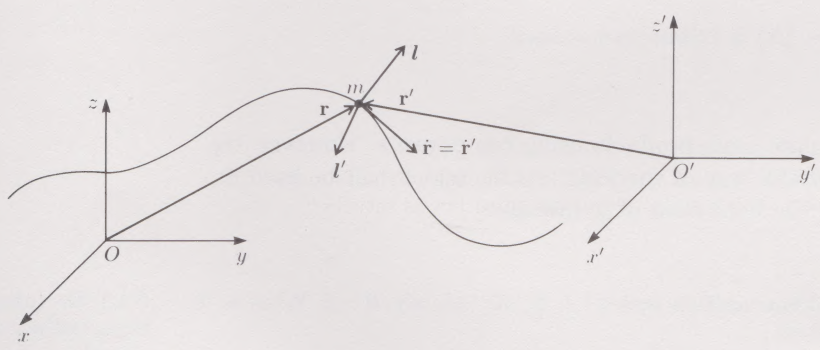


Figure 6

> **Angular momentum**
>
> For a particle which has linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$ and position vector $\mathbf{r}$ relative to an origin O, the *angular momentum* $\boldsymbol{l}$ relative to O is
>
> $$\boldsymbol{l} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\dot{\mathbf{r}}.$$

Example 1

A particle of mass m, on which no forces act, moves with constant speed u parallel to the y-axis in the (x, y)-plane. At time $t = t_0$ its path intersects the x-axis at $x = b$ (see Figure 7). Calculate the magnitude and direction of the angular momentum vector of the particle with respect to the origin O.

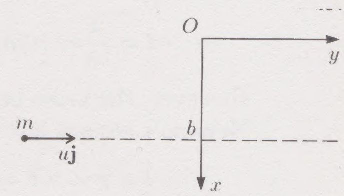


Figure 7

Solution

The position vector of the particle in the coordinate system of Figure 7 is

$$\mathbf{r} = b\mathbf{i} + u(t - t_0)\mathbf{j},$$

while the velocity vector is $\dot{\mathbf{r}} = u\mathbf{j}$. Thus the angular momentum vector is

$$\begin{aligned} \boldsymbol{l} &= \mathbf{r} \times m\dot{\mathbf{r}} \\ &= (b\mathbf{i} + u(t - t_0)\mathbf{j}) \times mu\mathbf{j} \\ &= bmu\mathbf{k}. \end{aligned}$$

Recall from *Unit 14* that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$.

Thus, even though the vector $\mathbf{r}$ is changing continuously, the cross product $\mathbf{r} \times m\dot{\mathbf{r}}$ is a constant vector. This vector points along the positive z-axis and has magnitude bmu.

The direction of the angular momentum in this example can also be found by applying the right hand rule to the situation shown in Figure 8 below. Pointing the thumb in the direction of $\mathbf{r}$ and the first finger in the direction of $m\dot{\mathbf{r}}$, the second finger, which gives the direction of $\boldsymbol{l}$, then points out of the page in the direction of $\mathbf{k}$.

For reasons of anatomical practicality, you may prefer to attempt this exercise by imagining the particle to have moved somewhat from its position in Figure 8!

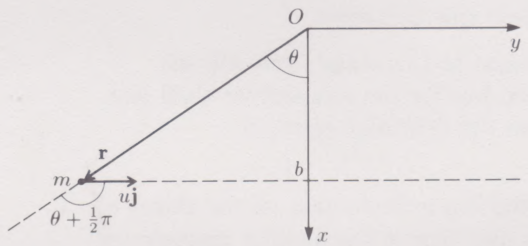


Figure 8

The magnitude of the angular momentum can be calculated alternatively by multiplying the product of the magnitudes of $\mathbf{r}$ and $m\dot{\mathbf{r}}$ by the sine of the angle between them. This gives

$$|\mathbf{r} \times m\dot{\mathbf{r}}| = |\mathbf{r}||m\dot{\mathbf{r}}| \sin\left(\theta + \tfrac{1}{2}\pi\right) = |\mathbf{r}|mu\cos\theta = bmu,$$

as before. □

In this unit we shall usually calculate cross products using components. However, the right hand rule often provides a useful way of checking results, and it will be used in Section 4 to explain qualitatively the behaviour of gyroscopes.

Exercise 3

At a given time, a particle of mass 3 has position vector $\mathbf{i} + 2\mathbf{j}$ and velocity $2\mathbf{i} - \mathbf{j}$. What is its angular momentum vector at this time?

[*Solution on page 48*]

As in the other mechanics texts of the course, all quantities are measured in SI units.

According to Newton's second law, the rate of change of the linear momentum of a single particle acted upon by a force $\mathbf{F}$ is given by

$$\dot{\mathbf{p}} = m\ddot{\mathbf{r}} = \mathbf{F}.$$

See *Unit 17* Subsection 3.1. We assume here that all particles have constant mass.

It turns out that a rather similar equation applies for the angular momentum $\boldsymbol{l}$. To obtain this equation, we differentiate both sides of Equation (1) with respect to time, which results in

See *Unit 14* Section 4.

$$\dot{\boldsymbol{l}} = \frac{d}{dt}(\mathbf{r} \times m\dot{\mathbf{r}}) = (\dot{\mathbf{r}} \times m\dot{\mathbf{r}}) + (\mathbf{r} \times m\ddot{\mathbf{r}}).$$

However, the cross product of two parallel vectors is zero, so that $\dot{\mathbf{r}} \times m\dot{\mathbf{r}} = \mathbf{0}$. Newton's second law therefore gives

$$\dot{\boldsymbol{l}} = \mathbf{r} \times m\ddot{\mathbf{r}} = \mathbf{r} \times \mathbf{F}.$$

The vector $\mathbf{r} \times \mathbf{F}$ was defined in *Unit 28* as the *torque* of the force $\mathbf{F}$ acting at the point with position vector $\mathbf{r}$, relative to the given origin. It will be denoted here by the symbol $\boldsymbol{\gamma}$. Notice that the torque

$$\boldsymbol{\gamma} = \mathbf{r} \times \mathbf{F} \tag{2}$$

involves the position vector $\mathbf{r}$ so that, just as for angular momentum, it depends on the choice of origin of the coordinate system. Nevertheless, for any *given* coordinate system, we have derived the *torque law* given below.

In *Unit 28* Subsection 4.1 we denoted the torque $\mathbf{r} \times \mathbf{F}$ by the symbol $\boldsymbol{\Gamma}$ (upper-case Greek gamma). Here we choose $\boldsymbol{\gamma}$ (lower-case Greek gamma) to represent the torque experienced by a single particle, because $\boldsymbol{\Gamma}$ will be employed later for the torque acting upon a system of particles.

Torque law for a single particle

The rate of change of a particle's angular momentum relative to a fixed point is equal to the applied torque relative to that point, that is,

$$\dot{\boldsymbol{l}} = \boldsymbol{\gamma}.$$

The torque law is as fundamental for the study of rotating bodies as is Newton's second law for the mechanical systems studied in previous units.

In the absence of applied torques (when $\boldsymbol{\gamma} = \mathbf{0}$), the torque law reduces to

$$\dot{\boldsymbol{l}} = \mathbf{0},$$

showing that the particle's angular momentum vector $\boldsymbol{l}$ is constant (or conserved). This result is known as the **law of conservation of angular momentum**.

Angular momentum conservation occurs when there are no forces acting on the particle. For instance, the particle considered in Example 1 was not subject to any forces. In accordance with Newton's first law, it travelled at constant speed in a straight line. The angular momentum vector of this particle was found to be

$$\boldsymbol{l} = bmu\mathbf{k},$$

which, in accordance with the conservation law, is constant in time.

However, even if the total force acting on the particle is non-zero, it is still possible for the torque to vanish and for the angular momentum conservation law to apply. This happens when the total force vector is permanently aligned with the position vector of the particle, which occurs if the particle is repelled from the origin O (as in Figure 9(i)) or attracted towards O (as in Figure 9(ii)).

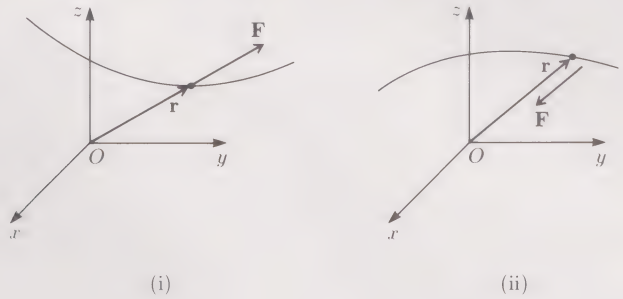


Figure 9

Then we have, respectively,

$$\mathbf{F} = F\hat{\mathbf{r}} \quad \text{or} \quad \mathbf{F} = -F\hat{\mathbf{r}},$$

where $F = |\mathbf{F}|$ and $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$ is a unit vector in the outward radial direction. Equation (2) then gives

$$\boldsymbol{\gamma} = \pm F\mathbf{r} \times \hat{\mathbf{r}} = \mathbf{0},$$

so that, by the torque law, $\boldsymbol{l}$ is a constant vector.

Law of conservation of angular momentum for a single particle

If the total torque acting on a particle about a fixed point is zero, then the angular momentum of the particle about the fixed point is constant. In other words, if $\boldsymbol{\gamma} = \mathbf{0}$ then $\boldsymbol{l}$ is constant.

Example 2

A particle of mass m at position $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ moves in a circle whose centre lies on the z-axis, with angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$. Find the angular momentum $\boldsymbol{l}$ of the particle in terms of x, y, z, ω, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

The angular velocity $\boldsymbol{\omega}$ of a particle in circular motion was introduced in *Unit 28* Subsection 4.3. It was shown there that if the origin lies on the axis of rotation then the velocity of the particle is given by

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}.$$

Solution

The motion of the particle is illustrated in Figure 10. By definition, we have $\boldsymbol{l} = \mathbf{r} \times m\dot{\mathbf{r}}$, and from *Unit 28* the velocity is

$$\begin{aligned}\dot{\mathbf{r}} &= \boldsymbol{\omega} \times \mathbf{r} = \omega\mathbf{k} \times \mathbf{r} \\ &= \omega\mathbf{k} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \omega(x\mathbf{j} - y\mathbf{i}).\end{aligned}$$

It follows that

$$\begin{aligned}\boldsymbol{l} &= \mathbf{r} \times m\omega(x\mathbf{j} - y\mathbf{i}) \\ &= m\omega(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times (x\mathbf{j} - y\mathbf{i}) \\ &= m\omega\left(-xz\mathbf{i} - yz\mathbf{j} + (x^2 + y^2)\mathbf{k}\right).\end{aligned}$$

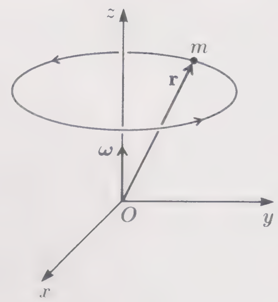


Figure 10

Note as a consequence of this result (which holds even when ω is not constant) that the direction of the angular momentum $\boldsymbol{l}$ is *not* in general the same as the direction of the angular velocity $\boldsymbol{\omega}$. □

Exercise 4

A particle of mass m moves in the (x, y)-plane around a circle of radius d whose centre is at the origin (see Figure 11). The angular velocity of the particle is $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$. Show that the angular momentum of the particle relative to the origin is $\boldsymbol{l} = md^2\dot{\theta}\mathbf{k}$.

[*Solution on page 48*]

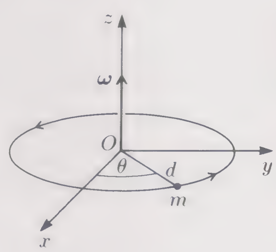


Figure 11

Example 3

A pendulum bob of mass m is tethered to the origin O by a light, taut string of constant length d. The bob swings to and fro in the (x, y)-plane (see Figure 12). Use the concepts of torque and angular momentum to derive the equation of motion of the bob.

Solution

The total force $\mathbf{F}$ acting on the bob is the vector sum of the forces due to gravity and to the tension T in the string, that is,

$$\mathbf{F} = mg\mathbf{i} + T(-\hat{\mathbf{r}}),$$

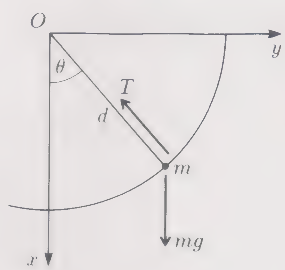


Figure 12

where $-\hat{\mathbf{r}}$ is the unit vector in the direction opposite to that of $\mathbf{r}$. The corresponding torque is

$$\begin{aligned}\boldsymbol{\gamma} = \mathbf{r} \times \mathbf{F} &= \mathbf{r} \times (mg\mathbf{i} - T\hat{\mathbf{r}}) \\ &= \mathbf{r} \times mg\mathbf{i} \qquad (\text{since } \mathbf{r} \times \hat{\mathbf{r}} = \mathbf{0}) \\ &= (x\mathbf{i} + y\mathbf{j}) \times mg\mathbf{i} \\ &= -mgy\mathbf{k} = -mgd\sin\theta\,\mathbf{k}.\end{aligned}$$

The angular momentum of the bob is given by the same expression as in Exercise 4, namely,

$$\boldsymbol{l} = md^2\dot{\theta}\mathbf{k}.$$

Hence the torque law gives

$$\frac{d}{dt}(md^2\dot{\theta}\mathbf{k}) = -mgd\sin\theta\,\mathbf{k}$$

or $$\ddot{\theta} = -\frac{g}{d}\sin\theta. \quad □$$

This result is the same as that obtained by different means in *Unit 15* Subsection 5.2 and *Unit 28* Subsection 3.3.

1.3 Angular momentum and torque for a many-particle system

We turn now to the major concern of this unit, which is to generalize the concepts of angular momentum and torque for a single particle, as discussed in the preceding subsection, to a system of many particles. The main goal here is to analyse the rotational motion of rigid bodies, which are many-particle systems of a special kind.

Consider a system of n particles, labelled $1, 2, \ldots, n$ as in Figure 13.

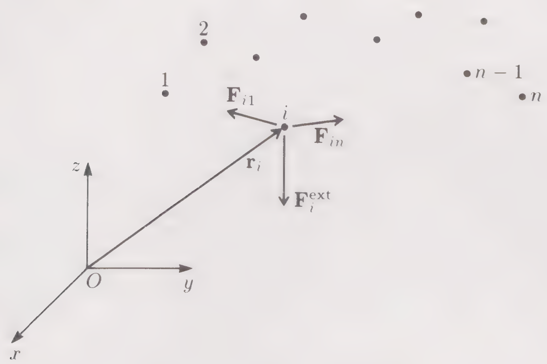


Figure 13

A typical particle, i, experiences inter-particle forces $\mathbf{F}_{i1}, \mathbf{F}_{i2}, \ldots, \mathbf{F}_{in}$ due to each of the other particles in the system, and a further force $\mathbf{F}_i^{\text{ext}}$ due to external influences, such as gravity. The total force acting on Particle i is therefore

$$\mathbf{F}_i = \mathbf{F}_{i1} + \mathbf{F}_{i2} + \cdots + \mathbf{F}_{in} + \mathbf{F}_i^{\text{ext}}.$$

The corresponding torque about the origin is

$$\begin{aligned}\boldsymbol{\gamma}_i &= \mathbf{r}_i \times \mathbf{F}_i \\ &= (\mathbf{r}_i \times \mathbf{F}_{i1}) + (\mathbf{r}_i \times \mathbf{F}_{i2}) + \cdots + (\mathbf{r}_i \times \mathbf{F}_{in}) + (\mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}) \\ &= \boldsymbol{\gamma}_{i1} + \boldsymbol{\gamma}_{i2} + \cdots + \boldsymbol{\gamma}_{in} + \boldsymbol{\gamma}_i^{\text{ext}},\end{aligned}$$

where we have introduced the notation

$$\boldsymbol{\gamma}_{ij} = \mathbf{r}_i \times \mathbf{F}_{ij} \qquad (i, j = 1, 2, \ldots, n;\ i \neq j)$$

to denote the *inter-particle torque* acting on Particle i due to the influence of Particle j, and

$$\boldsymbol{\gamma}_i^{\text{ext}} = \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}$$

to denote the torque on Particle i due to external forces. We next appeal to the strong form of Newton's third law, as described in Subsection 1.1, to show that $\boldsymbol{\gamma}_{ij} + \boldsymbol{\gamma}_{ji} = \mathbf{0}$. From the definition of $\boldsymbol{\gamma}_{ij}$ above, we have

$$\begin{aligned}\boldsymbol{\gamma}_{ij} + \boldsymbol{\gamma}_{ji} &= \mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji} \\ &= \mathbf{r}_i \times \mathbf{F}_{ij} - \mathbf{r}_j \times \mathbf{F}_{ij},\end{aligned}$$

since $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ even by the weak form of the third law. It follows that

$$\boldsymbol{\gamma}_{ij} + \boldsymbol{\gamma}_{ji} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}.$$

But, by the strong form of the third law, $\mathbf{F}_{ij}$ is *parallel* to $\mathbf{r}_i - \mathbf{r}_j$ (see Figure 14). Hence $(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = \mathbf{0}$, and it has been shown that

$$\boldsymbol{\gamma}_{ij} + \boldsymbol{\gamma}_{ji} = \mathbf{0}. \tag{3}$$

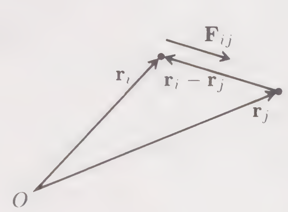


Figure 14

We now write down the torque law $\dot{\boldsymbol{l}}_i = \boldsymbol{\gamma}_i$ for each of the n particles of the system in turn, obtaining

$$\left.\begin{aligned}
\dot{\boldsymbol{l}}_1 &= \phantom{\boldsymbol{\gamma}_{21}} \quad \boldsymbol{\gamma}_{12} + \boldsymbol{\gamma}_{13} + \cdots + \boldsymbol{\gamma}_{1n} + \boldsymbol{\gamma}_1^{\text{ext}},\\
\dot{\boldsymbol{l}}_2 &= \boldsymbol{\gamma}_{21} \qquad\quad + \boldsymbol{\gamma}_{23} + \cdots + \boldsymbol{\gamma}_{2n} + \boldsymbol{\gamma}_2^{\text{ext}},\\
\dot{\boldsymbol{l}}_3 &= \boldsymbol{\gamma}_{31} + \boldsymbol{\gamma}_{32} \qquad\quad + \cdots + \boldsymbol{\gamma}_{3n} + \boldsymbol{\gamma}_3^{\text{ext}},\\
&\vdots \qquad \vdots \qquad\qquad\qquad\qquad \vdots\\
\dot{\boldsymbol{l}}_n &= \boldsymbol{\gamma}_{n1} + \boldsymbol{\gamma}_{n2} + \boldsymbol{\gamma}_{n3} + \cdots \qquad\quad + \boldsymbol{\gamma}_n^{\text{ext}}.
\end{aligned}\right\} \tag{4}$$

Note that there are no terms of the form $\boldsymbol{\gamma}_{ii}$ on the right-hand sides, since there are no inter-particle forces $\mathbf{F}_{ii}$.

The sum of the left-hand sides of these equations is

$$\dot{\boldsymbol{l}}_1 + \dot{\boldsymbol{l}}_2 + \dot{\boldsymbol{l}}_3 + \cdots + \dot{\boldsymbol{l}}_n = \sum_{i=1}^{n} \dot{\boldsymbol{l}}_i,$$

which must equal the sum of all the terms on the right-hand sides. Corresponding to every inter-particle torque $\boldsymbol{\gamma}_{ij}$ in the sum of the right-hand sides there is also a torque $\boldsymbol{\gamma}_{ji}$. (In fact, if $\boldsymbol{\gamma}_{ij}$ appears above the 'diagonal of blanks' in Equations (4) then $\boldsymbol{\gamma}_{ji}$ appears in the symmetrically opposite position below, and vice versa.) Thus the sum of the inter-particle torques is made up entirely of pairs of the form $\boldsymbol{\gamma}_{ij} + \boldsymbol{\gamma}_{ji}$, and by Equation (3) each such pair is equal to the zero vector. The sum of the right-hand sides is therefore just the sum of the external torques,

The argument which follows is similar to that used while deriving the motion of the centre of mass of an n-particle system in *Unit 17* Subsection 2.3.

$$\boldsymbol{\gamma}_1^{\text{ext}} + \boldsymbol{\gamma}_2^{\text{ext}} + \boldsymbol{\gamma}_3^{\text{ext}} + \cdots + \boldsymbol{\gamma}_n^{\text{ext}} = \sum_{i=1}^{n} \boldsymbol{\gamma}_i^{\text{ext}}.$$

Equating the sum of the left-hand sides of Equations (4) with the sum of the right-hand sides, we conclude that

$$\sum_{i=1}^{n} \dot{\boldsymbol{l}}_i = \sum_{i=1}^{n} \boldsymbol{\gamma}_i^{\text{ext}}. \tag{5}$$

This result can be rephrased by making two definitions, as follows.

1. The **total angular momentum L** of a system of particles, relative to a given origin, is defined to be the vector sum of the angular momenta of the individual particles, that is,

$$\mathbf{L} = \sum_{i=1}^{n} \boldsymbol{l}_i = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \dot{\mathbf{r}}_i). \tag{6}$$

The plural of 'momentum' is 'momenta'.

2. The **total external torque $\boldsymbol{\Gamma}^{\text{ext}}$** acting on a system of particles, relative to a given origin, is defined to be the vector sum of the external torques which act upon the individual particles, that is,

$$\boldsymbol{\Gamma}^{\text{ext}} = \sum_{i=1}^{n} \boldsymbol{\gamma}_i^{\text{ext}}. \tag{7}$$

We can now rewrite Equation (5) using the terms defined by Equations (6) and (7), to obtain the following important result.

Torque law for a system of particles

For a system of particles, the rate of change of the total angular momentum about the origin is equal to the total external torque about the origin, that is,

$$\dot{\mathbf{L}} = \boldsymbol{\Gamma}^{\text{ext}}. \tag{8}$$

In particular, if the total external torque is zero then the total angular momentum **L** is constant. This is the **law of conservation of angular momentum** for a system of particles.

This result is based on a specific assumption about the inter-particle torques, namely that $\boldsymbol{\gamma}_{ij} = -\boldsymbol{\gamma}_{ji}$, which is a consequence of the strong form of Newton's third law. In the real world, most of the 'classical' forces between particles satisfy this strong form.

An exception is the magnetic force between electrically-charged particles.

Example 4

Two particles connected by a straight spring are in motion on a smooth horizontal table, simultaneously rotating anticlockwise about their static centre of mass and vibrating. Use the torque law for a system of particles to show that the angular speed ω of the particles and the distance d between them are related by an equation of the form

$$d^2\omega = \text{constant}$$

throughout the motion.

Solution

Since the surface on which the particles move is both smooth and horizontal, there are no frictional forces and the effects of gravity are precisely balanced by the normal reaction forces. The only other forces acting on the particles are due to the extension or compression of the spring. These are inter-particle forces, satisfying the strong form of Newton's third law. Hence there is no net external force acting on either particle, and the total external torque on the system is therefore zero. It follows from the torque law for the two-particle system that $\dot{\mathbf{L}} = \mathbf{0}$, showing that the total angular momentum $\mathbf{L}$ is constant.

Suppose that the particles have masses m_1, m_2, and that their distances from the centre of mass O are respectively d_1 and d_2, as shown in Figure 15. If the polar angles of the particles are θ_1, θ_2, then $\dot{\theta}_1 = \dot{\theta}_2 = \omega$. Particle 1 has position vector

$$\mathbf{r}_1 = d_1\mathbf{e}_r$$

and velocity

$$\dot{\mathbf{r}}_1 = \dot{d}_1\mathbf{e}_r + d_1\dot{\mathbf{e}}_r = \dot{d}_1\mathbf{e}_r + d_1\omega\mathbf{e}_\theta,$$

since $\dot{\theta}_1 = \omega$. Hence the angular momentum of Particle 1 is

$$\begin{aligned}\boldsymbol{l}_1 &= \mathbf{r}_1 \times m_1\dot{\mathbf{r}}_1\\ &= d_1\mathbf{e}_r \times m_1(\dot{d}_1\mathbf{e}_r + d_1\omega\mathbf{e}_\theta)\\ &= m_1 d_1^2\omega\mathbf{e}_r \times \mathbf{e}_\theta = m_1 d_1^2\omega\mathbf{k}.\end{aligned}$$

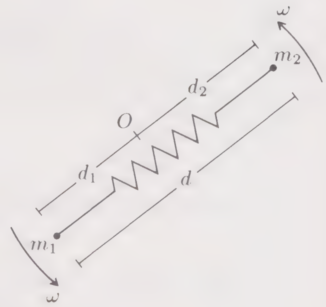


Figure 15

This shows that the result of Exercise 4 can be extended from motion in a circle to any planar motion.

Similarly, Particle 2 has angular momentum $\boldsymbol{l}_2 = m_2 d_2^2\omega\mathbf{k}$, so that the total angular momentum of the system is

$$\mathbf{L} = (m_1 d_1^2 + m_2 d_2^2)\omega\mathbf{k}.$$

Now, by the definition of centre of mass, we have

$$m_1 d_1 = m_2 d_2.$$

The centre of mass of a two-particle system was defined in *Unit 17* Subsection 2.2.

Since also $d_1 + d_2 = d$, we find that

$$d_1 = \frac{m_2 d}{m_1 + m_2}, \qquad d_2 = \frac{m_1 d}{m_1 + m_2}.$$

Thus the constant magnitude L of the angular momentum vector is

$$\begin{aligned}L &= (m_1 d_1^2 + m_2 d_2^2)\omega\\ &= \frac{(m_1 m_2^2 + m_2 m_1^2)d^2\omega}{(m_1 + m_2)^2}\\ &= \frac{m_1 m_2}{m_1 + m_2} d^2\omega.\end{aligned}$$

Since the masses of the particles are constant, this establishes that the quantity $d^2\omega$ is constant throughout the motion. □

Exercise 5

A system of particles moves under gravity, which acts in the negative z-direction. Show that the z-component of the total angular momentum is constant.

Exercise 6

A binary star system consists of two stars, of masses m_1 and m_2, rotating about their centre of mass with a constant separation distance d. Express the period of revolution τ of the system in terms of the quantities above and the magnitude L of the total angular momentum.

[*Solutions on page 48*]

We next ask you to verify a result which was used without proof in *Unit 28*, namely, that the total external torque acting on a static system is zero.

Exercise 7

Show that the total external torque acting on a static system of particles is zero.

[*Solution on page 49*]

The table below, which concludes this section, gives a comparison between the concepts involved with linear momentum (*Unit 17*) and with angular momentum.

Table 1

Linear momentum		**Angular momentum**	
Single particle			
Linear momentum	$\mathbf{p} = m\dot{\mathbf{r}}$	Angular momentum	$\boldsymbol{l} = \mathbf{r} \times m\dot{\mathbf{r}}$
Force	$\mathbf{F}$	Torque	$\boldsymbol{\gamma} = \mathbf{r} \times \mathbf{F}$
Newton's second law is	$\dot{\mathbf{p}} = \mathbf{F}$	Newton's second law leads to	$\dot{\boldsymbol{l}} = \boldsymbol{\gamma}$
In the absence of forces, linear momentum is conserved.		In the absence of torques, angular momentum is conserved.	
System of n particles			
Total linear momentum	$\mathbf{P} = \sum_{i=1}^{n} m_i\dot{\mathbf{r}}_i$	Total angular momentum	$\mathbf{L} = \sum_{i=1}^{n}(\mathbf{r}_i \times m_i\dot{\mathbf{r}}_i)$
Total external force	$\mathbf{F}^{\text{ext}} = \sum_{i=1}^{n} \mathbf{F}_i^{\text{ext}}$	Total external torque	$\boldsymbol{\Gamma}^{\text{ext}} = \sum_{i=1}^{n} \left(\mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}\right)$
Newton's second and third laws lead to	$\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}}$	Newton's second and third (strong form) laws lead to	$\dot{\mathbf{L}} = \boldsymbol{\Gamma}^{\text{ext}}$
With zero total external force, the total linear momentum is conserved.		With zero total external torque, the total angular momentum is conserved.	

Summary of Section 1

1. The **strong form of Newton's third law** states that for a pair of interacting particles, the force $\mathbf{F}_{12}$ on Particle 1 due to Particle 2 is equal in magnitude but opposite in direction to the force $\mathbf{F}_{21}$ on Particle 2 due to Particle 1, that is,

   $$\mathbf{F}_{12} = -\mathbf{F}_{21},$$

   and each of these forces acts along the line which joins the two particles.

2. For a particle which has position vector $\mathbf{r}$ relative to an origin O and linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$, the **angular momentum $\boldsymbol{l}$** relative to O is

   $$\boldsymbol{l} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\dot{\mathbf{r}}.$$

3. The **torque law** for a single particle states that the rate of change of the particle's angular momentum relative to a fixed point is equal to the applied torque relative to that point, that is,

   $$\dot{\boldsymbol{l}} = \boldsymbol{\gamma}.$$

   In the special case where the total torque acting on the particle is zero, the angular momentum of the particle about the fixed point is constant. In other words, if $\boldsymbol{\gamma} = \mathbf{0}$ then $\boldsymbol{l}$ is constant. This is the **law of conservation of angular momentum** for a single particle.

4. The **total angular momentum L** of a system of n particles, relative to a given origin, is the vector sum of the angular momenta of the individual particles, that is,

$$\mathbf{L} = \sum_{i=1}^{n} \boldsymbol{l}_i = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \dot{\mathbf{r}}_i),$$

where $\mathbf{r}_i$ is the position vector of the ith particle and m_i is its mass.

5. The **total external torque $\boldsymbol{\Gamma}^{\text{ext}}$** acting on a system of n particles, relative to a given origin, is defined to be the vector sum of the external torques which act upon the individual particles, that is,

$$\boldsymbol{\Gamma}^{\text{ext}} = \sum_{i=1}^{n} \boldsymbol{\gamma}_i^{\text{ext}}.$$

6. The **torque law** for a system of particles states that the rate of change of the total angular momentum about the origin is equal to the total external torque about the origin, that is,

$$\dot{\mathbf{L}} = \boldsymbol{\Gamma}^{\text{ext}}.$$

This law applies to any system for which the inter-particle forces satisfy the strong form of Newton's third law.

In particular, if the total external torque is zero then the total angular momentum $\mathbf{L}$ is constant. This is the **law of conservation of angular momentum** for a system of particles.

2 Rigid-body rotation

2.1 Rotation about a fixed axis

The torque law $\dot{\mathbf{L}} = \boldsymbol{\Gamma}^{\text{ext}}$ for a system of particles is very general. It is based on the assumption that the inter-particle torques cancel, but makes no other assumptions about the particles or the way in which they move. However, in this section we specialize by concentrating upon application of the torque law to *rigid* objects which rotate about fixed axes.

A rigid body, as its name suggests, is an object that retains the same size and shape no matter how it moves or what forces it experiences. This is an idealization of the real world, since no physical object is completely rigid. For example, high-speed photographs of snooker balls (which are fairly rigid) show that they deform at the moment of impact. Nevertheless, many objects do behave as if they were *approximately* rigid, so that the idealization of a rigid body provides a sensible mathematical model. We shall adopt the strategy of *Unit 17*, and regard any solid object as being a collection of many particles. The precise definition of a rigid body is then as follows.

This definition was given in *Unit 28* Subsection 2.3, where static rigid bodies were considered.

> A **rigid body** is a many-particle system with the property that all the inter-particle distances remain constant in time.

A rigid body cannot flex or vibrate, but it can still move in a variety of ways. We now restrict attention to a special type of motion, called **rigid-body rotation**, in which a line of points in the body (known as the **axis of rotation**) remains fixed, while other points move around this line. Figure 1 (overleaf) represents such a motion where, for simplicity, the z-axis of the coordinate system has been chosen as the axis of rotation.

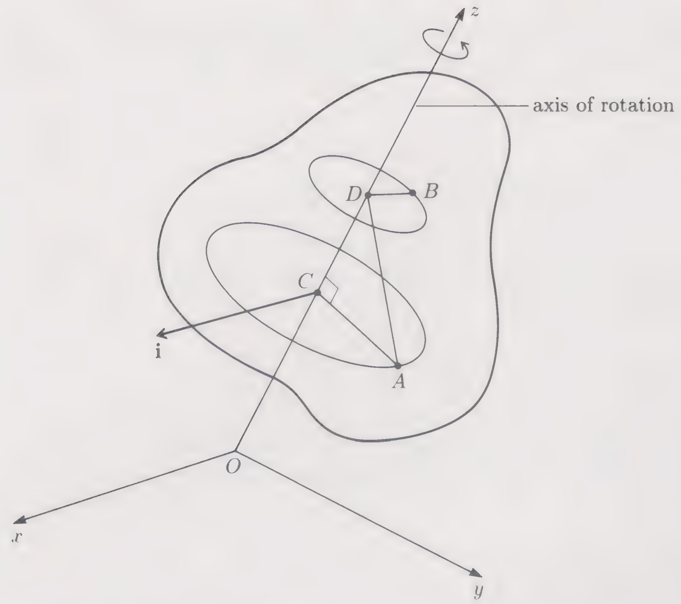


Figure 1

Particles which lie on the axis of rotation, such as C and D, do not move. However, Particle A does move, keeping a constant distance from each of C and D. This means that Particle A is confined to a circle centred on, and perpendicular to, the z-axis. Using the result of Example 2 of Section 1, the velocity of Particle A is therefore given by $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$, where $\mathbf{r} = \mathbf{r}_A$, $\boldsymbol{\omega} = \dot{\theta}_A\mathbf{k}$ and θ_A is the angle between the radius CA and the unit vector $\mathbf{i}$. In other words, we have

$$\dot{\mathbf{r}}_A = \dot{\theta}_A(\mathbf{k} \times \mathbf{r}_A).$$

Similarly, the velocity of Particle B is

$$\dot{\mathbf{r}}_B = \dot{\theta}_B(\mathbf{k} \times \mathbf{r}_B),$$

where θ_B is the angle between the radius DB and the unit vector $\mathbf{i}$.

Now the rigidity of the body leads to a considerable simplification: Particles A and B rotate about the z-axis at the same angular rate, so that

$$\dot{\theta}_A = \dot{\theta}_B.$$

In fact, all of the particles in the body share this same angular motion about the z-axis (except, of course, for the motionless particles which actually lie on the z-axis). This fact allows us to omit the subscripts from $\dot{\theta}_A, \dot{\theta}_B, \ldots$, and to regard $\dot{\theta}$ as the rate of rotation of the rigid body as a whole. We write

$$\dot{\theta} = \dot{\theta}_A = \dot{\theta}_B = \ldots$$

and $$\boldsymbol{\omega} = \dot{\theta}\mathbf{k},$$

where $\boldsymbol{\omega}$ is the **angular velocity of the rigid body**. The velocity of the ith particle in the body can be expressed in terms of $\boldsymbol{\omega}$ or $\dot{\theta}$ by

We introduced $\boldsymbol{\omega}$ in *Unit 28* Subsection 4.3 as the angular velocity of a single particle.

$$\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \mathbf{r}_i = \dot{\theta}\mathbf{k} \times \mathbf{r}_i. \qquad (1)$$

Note that these equations apply even to stationary particles on the axis of rotation, since for such particles $\mathbf{r}_i$ is parallel to $\mathbf{k}$.

The total angular momentum $\mathbf{L}$ of the rigid body about the given origin can then be related to $\dot{\theta}$. From Equation (6) of Section 1 and Equation (1), we have

$$\mathbf{L} = \sum_{i=1}^{n}(\mathbf{r}_i \times m_i\dot{\mathbf{r}}_i) = \sum_{i=1}^{n}\left(\mathbf{r}_i \times m_i\dot{\theta}(\mathbf{k} \times \mathbf{r}_i)\right).$$

Repetition of the calculation of Example 2 of Section 1 for each particle then leads to the equation

$$\mathbf{L} = \dot{\theta}\sum_{i=1}^{n} m_i\left(-x_iz_i\mathbf{i} - y_iz_i\mathbf{j} + (x_i^2 + y_i^2)\mathbf{k}\right).$$

This expression is complicated because $\mathbf{L}$ may have x- and y-components: in general, the total angular momentum vector has components *perpendicular* to the axis of

rotation (as noted in Example 2 of Section 1 for the single-particle case). However, this complication is often irrelevant. In many applications we need consider only the component of the total angular momentum along the axis of rotation. Indeed, for a body symmetrical about the axis of rotation (as in Figure 2) this is the only non-zero component, as you will see in Section 4. For a rigid body rotating about the z-axis (as in Figure 1) the z-component of the total angular momentum $\mathbf{L}$ is given by

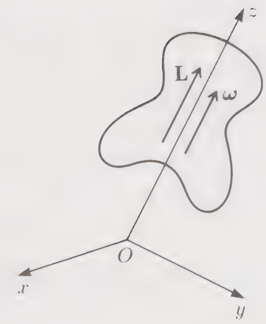


Figure 2

$$L_z = \dot{\theta}\sum_{i=1}^{n} m_i\left(x_i^2 + y_i^2\right) = \dot{\theta}\sum_{i=1}^{n} m_i d_i^2,$$

where $d_i = \sqrt{x_i^2 + y_i^2}$ is the distance of the ith particle from the axis of rotation. Note that d_i is constant in time, unlike x_i and y_i. We now recall that the **moment of inertia** I of the body about the z-axis is defined to be

$$I = \sum_{i=1}^{n} m_i d_i^2. \tag{2}$$

The moment of inertia I was defined in *Unit 27* Subsection 2.1. Since m_i and d_i are constant for each particle, it follows that I is also constant.

The moment of inertia for a continuous rigid body is calculated by replacing the finite sum in Equation (2) by an integral, as discussed in the next subsection.

We then have

$$L_z = I\dot{\theta}. \tag{3}$$

Combining this with the torque law for a system of particles, we obtain the following important result.

> **Torque law for a rigid body rotating about a fixed axis**
>
> If a rigid body rotates around the (fixed) z-axis, with angular velocity $\dot{\theta}\mathbf{k}$, then the rate of change of $\dot{\theta}$ is given by
>
> $$I\ddot{\theta} = \frac{d}{dt}(I\dot{\theta}) = \dot{L}_z = \Gamma_z^{\text{ext}}, \tag{4}$$
>
> where I is the moment of inertia of the rigid body about the axis of rotation and $L_z, \Gamma_z^{\text{ext}}$ are the z-components of the total angular momentum and external torque acting on the body about the origin, respectively.

Here $\Gamma_z^{\text{ext}} = \mathbf{\Gamma}^{\text{ext}} \cdot \mathbf{k}$, where $\mathbf{k}$ is the unit vector in the z-direction.

Equation (4) may be regarded as the analogue, for rigid-body rotation, of the z-component of Newton's second law for a system of particles,

$$M\ddot{Z} = \frac{d}{dt}(M\dot{Z}) = \dot{P}_z = F_z^{\text{ext}},$$

where M is the total mass of the system, Z is the z-component of the position of its centre of mass, and P_z, F_z^{ext} are the z-components of the total linear momentum and external force, respectively.

Finally, note that we can write the kinetic energy of the rigid body in terms of its moment of inertia about the axis of rotation. Summing the kinetic energies of the individual particles, the total kinetic energy is

$$T = \sum_{i=1}^{n} \tfrac{1}{2} m_i |\dot{\mathbf{r}}_i|^2 = \tfrac{1}{2}\sum_{i=1}^{n} m_i (d_i\dot{\theta})^2 = \tfrac{1}{2} I\dot{\theta}^2. \tag{5}$$

The result $|\dot{\mathbf{r}}_i| = d_i|\dot{\theta}|$ is used here. This follows because the motion of each particle is circular, so that the formulas of *Unit 28* Subsection 3.1 apply.

When dealing with the motion of a rigid body, the concept of moment of inertia is indispensable. This depends on the mass distribution in the body and on the axis to which it is referred, but it encapsulates in a single scalar quantity the information needed to describe rotation about the axis.

Exercise 1

Figure 3 shows a rigid body consisting of two balls of mass m connected by a light bar of length $2d$.

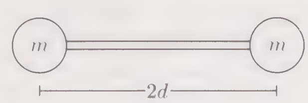


Figure 3

(i) Assuming that the two balls can be modelled by particles, find the moment of inertia of the body about an axis perpendicular to the page which passes through the centre of the bar.

(ii) Suppose that the body rotates about its centre in the plane of the page, with angular speed $\dot{\theta}$. Use Equation (5) to find its kinetic energy.

Exercise 2

A lift containing miners, of combined mass M, is lowered down a mine-shaft to a depth h by means of a cable running around a heavy flywheel (see Figure 4). The flywheel has radius R and moment of inertia I about its axis of rotation. Neglecting friction, and assuming that the motion starts from rest, use conservation of mechanical energy to find the speed v of the lift as it hits the bottom of the shaft.

[*Solutions on page 49*]

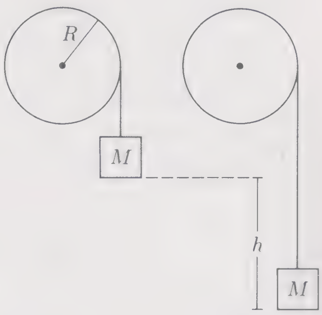


Figure 4

2.2 Finding moments of inertia

In order to predict the motion of a rotating rigid body, we need to know its moment of inertia I about the axis of rotation. This can be seen from Equation (4) or, in cases where energy conservation is to be used, from Equation (5). The purpose of this subsection is to demonstrate how to obtain the moments of inertia of some continuous rigid bodies. The television section of *Unit 27* showed how these may be calculated, by converting a sum over many particles into a volume integral. That is, the sum

$$I = \sum_{i=1}^{n} m_i d_i^2 \tag{2}$$

is replaced by the integral

$$I = \int_B \rho(\mathbf{r}) d^2(\mathbf{r})\, dV, \tag{6}$$

where $\rho(\mathbf{r})$ is the density of mass at the point $\mathbf{r}$, $d(\mathbf{r})$ is the distance of the point $\mathbf{r}$ from the axis of rotation, and the integration is taken over the volume of the rigid body B. For a homogeneous object, the density has a constant value, ρ_0 say, in which case Equation (6) becomes

$$I = \rho_0 \int_B d^2(\mathbf{r})\, dV.$$

Comparing this with the total mass,

$$M = \int_B \rho(\mathbf{r})\, dV = \rho_0 \int_B dV,$$

we obtain the expression

$$I = \frac{M \int_B d^2(\mathbf{r})\, dV}{\int_B dV} \tag{7}$$

for the moment of inertia of a homogeneous rigid object. However, in most cases it is possible, and simpler, to apply Equation (2) or (6) rather than Equation (7).

Example 1 (Linear rod)

A uniform rod of length h and mass M has negligible dimensions in directions perpendicular to its length. It lies along the x-axis with the origin at its centre. Find its moment of inertia about the z-axis.

Solution

According to the definition in Equation (2), the moment of inertia is

$$I = \sum_i m_i d_i^2,$$

where m_i is the mass of the ith particle, d_i is its distance from the axis of rotation, and the sum is taken over all particles making up the object. For continuous bodies, such as the rod, an unbounded number of particles is involved, so we proceed by first thinking of the body as made up of a finite number of small portions of material, each of which is represented by a particle, and then take the limit of the resulting sum as the number of portions increases and their size tends to zero.

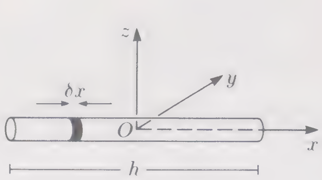


Figure 5

Consider the small portion of rod shown in Figure 5, which is at position x and has width δx. The rod has uniform linear density $\rho_0 = M/h$, so that the portion being

considered has mass $\rho_0 \delta x$. Its distance from the axis of rotation is $|x|$, so that I is equal to the sum of terms

$$\rho_0 x^2 \delta x,$$

over all such portions of rod, in the limit as δx tends to zero. Thus the moment of inertia is given by the integral

$$I = \int_{-h/2}^{h/2} \rho_0 x^2 \, dx = \frac{2\rho_0}{3}\left(\frac{h}{2}\right)^3 .$$

Since $\rho_0 = M/h$, we conclude that the moment of inertia of the rod is

$$I = \tfrac{1}{12} M h^2. \quad \square$$

Exercise 3

The rod described in Example 1 is now placed along the x-axis with the origin at one end (see Figure 6). Find its moment of inertia about the z-axis in this new position.

[*Solution on page 49*]

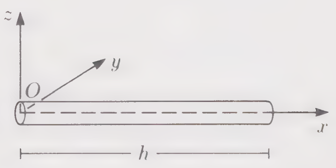


Figure 6

Example 2 (Disc)

A uniform disc of negligible thickness has radius R and mass M. Calculate its moment of inertia about an axis perpendicular to the disc and through its centre.

Solution

Suppose that the axis of rotation is the z-axis, so that the disc lies in the (x, y)-plane, as shown in Figure 7. We shall obtain the moment of inertia I of the disc by evaluating a surface integral using polar coordinates, as in *Unit 27*.

Unit 27 Subsection 2.3

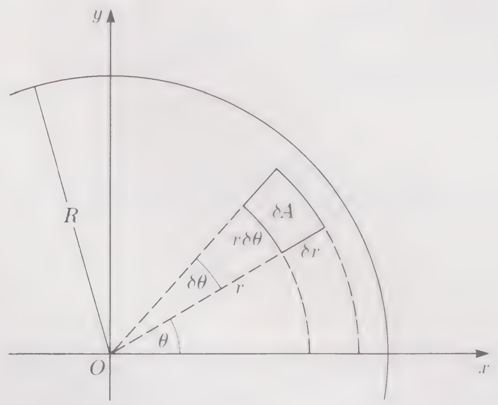


Figure 7

The disc has a two-dimensional (surface) density $\rho_0 = M/(\pi R^2)$, so that the small section of area

$$\delta A = r \, \delta r \, \delta\theta$$

shown in Figure 7 has mass $\rho_0 \, \delta A$. This section is a distance r from the z-axis, so that its contribution to the moment of inertia is

$$(\rho_0 r \, \delta r \, \delta\theta) r^2 = \rho_0 r^3 \, \delta r \, \delta\theta.$$

The surface integral for I is therefore

$$I = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \rho_0 r^3 \, dr \, d\theta = 2\pi\rho_0 \int_{r=0}^{R} r^3 \, dr = \tfrac{1}{2}\pi\rho_0 R^4.$$

On putting $\rho_0 = M/(\pi R^2)$, we have

$$I = \tfrac{1}{2} M R^2. \quad \square$$

Exercise 4

A uniform hoop of negligible thickness has mass M and radius R. Find its moment of inertia about an axis through its centre and perpendicular to its plane (that is, about the z-axis if the hoop is positioned as in Figure 8).

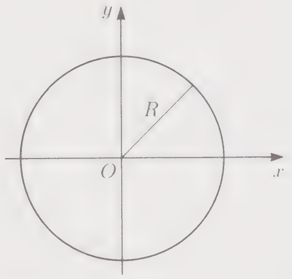


Figure 8

Exercise 5

A uniform annular lamina of negligible thickness, with mass M, inner radius a and outer radius R, lies in the (x, y)-plane with its centre at the origin, as shown in Figure 9. Find the moment of inertia of the lamina about the z-axis.

[*Solutions on page 49*]

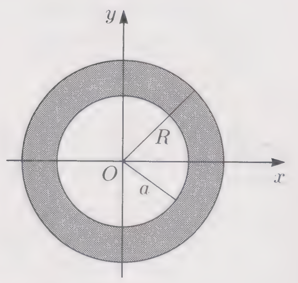


Figure 9

Example 3 (Cube)

A uniform cube of side a and mass M is placed with three of its sides along the coordinate axes, as shown in Figure 10. Find the moment of inertia of the cube about the z-axis.

Solution

We use Equation (6) with mass density $\rho_0 = M/a^3$, volume element $\delta V = \delta x\,\delta y\,\delta z$ and (distance)^2 from the z-axis given by $d^2(\mathbf{r}) = x^2 + y^2$. Thus the moment of inertia is

$$\begin{aligned}
I &= \int_B \rho(\mathbf{r}) d^2(\mathbf{r})\, dV \\
&= \frac{M}{a^3} \int_{z=0}^{a} \int_{y=0}^{a} \int_{x=0}^{a} \left(x^2 + y^2\right)\, dx\, dy\, dz \\
&= \frac{M}{a^2} \int_{y=0}^{a} \int_{x=0}^{a} \left(x^2 + y^2\right)\, dx\, dy \\
&= \frac{M}{a^2} \int_{x=0}^{a} \left(ax^2 + \tfrac{1}{3}a^3\right)\, dx \\
&= \tfrac{2}{3} Ma^2. \quad \square
\end{aligned}$$

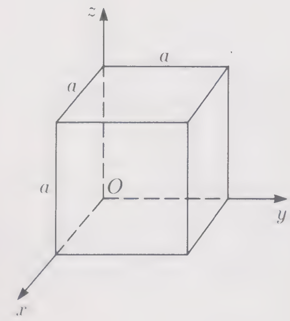


Figure 10

Exercise 6

Figure 11 shows a homogeneous rigid rectangular brick of mass M.

(i) The axis AA' passes through the centres of the two faces with sides of lengths a and b. Show that the moment of inertia of the brick about this axis is

$$\tfrac{1}{12} M\left(a^2 + b^2\right).$$

(ii) The axis BB' lies in the surface of a face whose edges have lengths b and c, and bisects two edges of length b. Show that the moment of inertia of the brick about this axis is

$$\tfrac{1}{12} M\left(4a^2 + b^2\right).$$

[*Solution on page 49*]

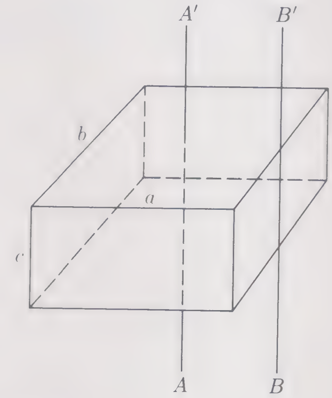


Figure 11

Table 1 (see next two pages) contains for reference the moments of inertia of some solid rigid bodies about axes through their mass centres. Note that the moments of inertia of some uniform laminas (rigid bodies of negligible thickness) may be deduced as limiting cases of the formulas given in Table 1. Thus the outcome of Exercise 5 is obtained from the fourth row of the table by considering the annular region defined in the exercise to be a hollow cylinder of negligible length. Similarly, the moment of inertia $\frac{2}{3}MR^2$ for a spherical shell of radius R is found by taking the limiting case of a hollow sphere as the thickness tends to zero (as indicated in the second row of the table).

Table 1: Moments of inertia of homogeneous rigid bodies

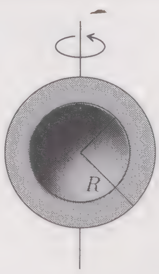
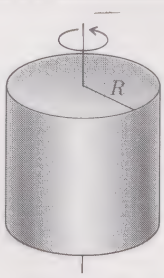
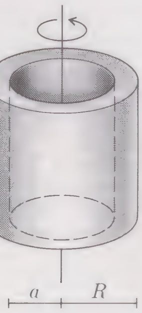
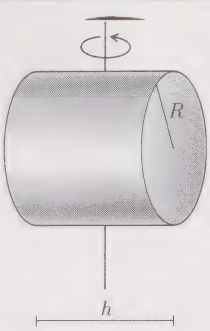
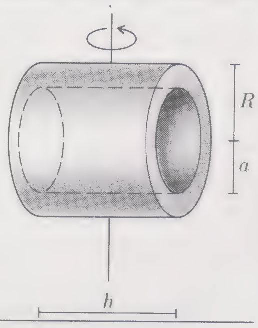
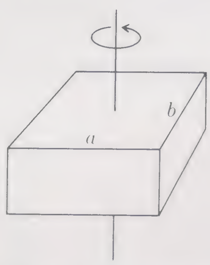
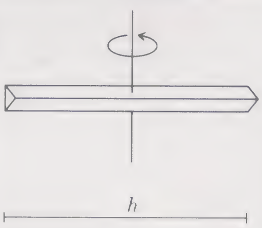
Homogeneous rigid body of mass M; axis	Diagram	Dimensions	Moment of inertia about given axis
solid sphere; axis through centre of mass		radius R	$\frac{2}{5}MR^2$
hollow sphere; axis through centre of mass		inner radius a, outer radius R	$\frac{2}{5}M\left(\frac{R^5-a^5}{R^3-a^3}\right)$ $\simeq \frac{2}{3}MR^2$ if $R \simeq a$
solid cylinder; axis through centre of mass and along axis of cylinder		radius R	$\frac{1}{2}MR^2$
hollow cylinder; axis through centre of mass and along axis of cylinder		inner radius a, outer radius R	$\frac{1}{2}M(R^2+a^2)$ $\simeq MR^2$ if $R \simeq a$
solid cylinder; axis through centre of mass and perpendicular to axis of cylinder		radius R, length h	$\frac{1}{4}MR^2 + \frac{1}{12}Mh^2$

Table 1 (continued)

Homogeneous rigid body of mass M; axis	Diagram	Dimensions	Moment of inertia about given axis
hollow cylinder; axis through centre of mass and perpendicular to axis of cylinder		inner radius a, outer radius R, length h	$\frac{1}{4}M(R^2+a^2)+\frac{1}{12}Mh^2$ $\simeq \frac{1}{2}MR^2+\frac{1}{12}Mh^2$ if $R \simeq a$
solid rectangular brick; axis through centre of mass and perpendicular to one pair of faces		faces perpendicular to axis have sides of lengths a and b	$\frac{1}{12}M(a^2+b^2)$
thin rod (cross-section of arbitrary shape); axis through centre of mass and perpendicular to rod		length h	$\frac{1}{12}Mh^2$

There is a useful result called the *parallel axes theorem* which permits the moment of inertia about an arbitrary axis to be deduced rapidly from knowledge of the moment of inertia about a parallel axis which passes through the centre of mass of the body. As a consequence, the information in Table 1 is of wider application than might at first be supposed. This theorem is stated below.

Parallel axes theorem

Suppose that a body of mass M has centre of mass at the point G, and that the fixed axis A is a distance d from G (see Figure 12(i)). Then the moment of inertia I of the body about the axis A is

$$I = I_G + Md^2, \tag{8}$$

where I_G is the moment of inertia of the body about an axis B which passes through G and is parallel to A (see Figure 12(ii)).

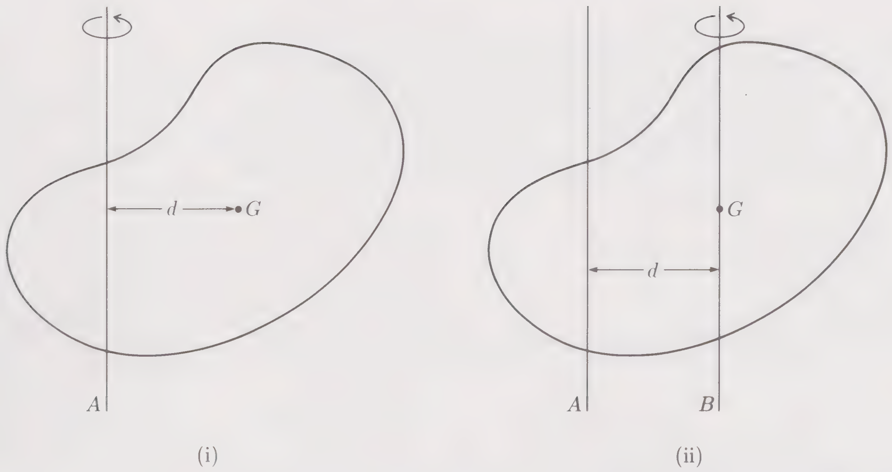


Figure 12

Proof of the parallel axes theorem

This proof is fairly straightforward. However, if you are at all short of time then you should omit the proof at this stage and concentrate on how the result may be applied.

Without loss of generality, the z-axis may be taken to coincide with the fixed axis A, with the y-axis chosen to intersect the parallel axis B, as shown in Figure 13. Then the centre of mass G has y-coordinate d. Since its position vector $\mathbf{R}$ is defined by

$$M\mathbf{R} = \sum m_i \mathbf{r}_i$$

(where here and below summations are taken over all particles i of the body), it follows on taking y-components that

$$Md = \sum m_i y_i. \tag{9}$$

This result will be useful shortly.

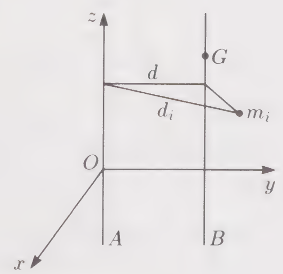


Figure 13

The particle indicated in Figure 13 is a distance d_i from the z-axis, where

$$d_i^2 = x_i^2 + y_i^2.$$

The square of its distance from the axis B is

$$x_i^2 + (y_i - d)^2,$$

so that the moments of inertia I, I_G about the axes A, B respectively are given by

$$I = \sum m_i(x_i^2 + y_i^2) \quad \text{and} \quad I_G = \sum m_i \left(x_i^2 + (y_i - d)^2\right).$$

We can therefore deduce that

$$\begin{aligned} I &= I_G + \sum 2m_i y_i d - \sum m_i d^2 \\ &= I_G + 2d \sum m_i y_i - d^2 \sum m_i. \end{aligned}$$

Using Equation (9) and the fact that $\sum m_i = M$, we obtain

$$I = I_G + Md^2,$$

and the theorem is proved.

End of proof

The parallel axes theorem expresses a moment of inertia I as the sum of two terms, each of which depends on the centre of mass. The term Md^2 can be thought of as a property *of* the centre of mass (in fact, it is the moment of inertia about axis A of a particle of mass M placed at the centre of mass), while the term I_G is defined *relative* to the centre of mass. In Section 3 you will see theorems of a similar nature which concern kinetic energy, angular momentum and torque in many-particle systems.

As an illustration of the use of Equation (8), consider once more the problem described in Exercise 6(ii) and illustrated in Figure 11 (page 20). In this case, we seek the moment of inertia about the axis BB', which is a distance $d = \frac{1}{2}a$ from the centre of mass. The moment of inertia I_G about the parallel axis AA' through the centre of mass is obtained from Table 1 or from Exercise 6(i). Then the parallel axes theorem gives

$$\begin{aligned} I &= I_G + Md^2 \\ &= \tfrac{1}{12}M(a^2+b^2) + M(\tfrac{1}{2}a)^2 \\ &= \tfrac{1}{12}M(4a^2+b^2), \end{aligned}$$

as before. Similar methods may be used to find all the moments of inertia required in this unit.

Exercise 7

Figure 14 shows a uniform disc of negligible thickness, with radius R and mass M. Use Table 1 and the parallel axes theorem to find the moment of inertia of the disc about the axis shown, which lies in the plane of the disc and is tangential to it.

[*Solution on page 49*]

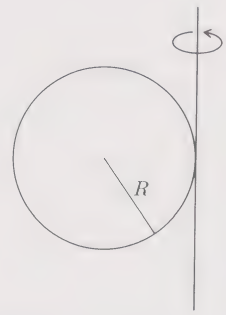


Figure 14

2.3 Conservation of angular momentum

Having spent the previous subsection calculating moments of inertia, we now revert to applications of the torque law for a system rotating about a fixed axis. If the axis of rotation is the z-axis, and the system is a rigid body, then the torque law takes the form

$$I\ddot{\theta} = \frac{d}{dt}(I\dot{\theta}) = \dot{L}_z = \Gamma_z^{\text{ext}}, \tag{4}$$

where I is the moment of inertia of the body about the z-axis, $\dot{\theta}\mathbf{k}$ is the angular velocity, and $L_z, \Gamma_z^{\text{ext}}$ are the z-components of the total angular momentum and external torque about the origin, respectively.

In the next subsection we shall investigate applications of this equation in which the total external torque has a non-zero z-component, but here we concentrate on cases where

$$\Gamma_z^{\text{ext}} = 0$$

and, as a consequence,

$$L_z = \text{constant}.$$

Actually, none of the systems to be considered in this subsection is a simple rigid body. Despite this we can still use the facts that

(i) if the z-component of the total external torque is $\Gamma_z^{\text{ext}} = 0$, then L_z is constant for the system as a whole;

(ii) for any subsystem which is a rigid body, or for the whole system during any period when it rotates like a rigid body, the angular momentum z-component is given as before by $I\dot{\theta}$, and the rotational kinetic energy is $\frac{1}{2}I\dot{\theta}^2$ (from Equation (5)).

This follows from the torque law for an arbitrary many-particle system (see Equation (8) of Section 1) or, if $\mathbf{\Gamma}^{\text{ext}} = \mathbf{0}$, from the law of conservation of angular momentum.

The following example shows how these rules may be applied.

Example 4

A man of mass m stands on a large uniform horizontal disc of mass M and radius R. The disc is free to rotate without friction about a vertical axis through its centre. The man stands initially at the edge of the disc, which rotates at a steady rate of $1\,\text{rad}\,\text{s}^{-1}$.

He then walks towards the middle of the disc until he reaches the centre. The man is to be modelled as a particle, and it may be assumed that the reaction forces at the axis of rotation produce no vertical component of torque.

(i) What is the angular speed of the disc when the man reaches the centre?

(ii) Show that the man must expend at least $\frac{1}{2}mR^2(1 + 2m/M)$ joules of energy in order to reach the centre.

Solution

(i) Figure 15 shows the disc, the man and the chosen coordinate system. The external forces acting on the man and the disc are due to gravity and the normal reaction at the centre of the disc. These forces do not produce a z-component of torque, so the z-component of the total external torque on the system of man-plus-disc is zero. Consequently, the z-component L_z of the total angular momentum of the system is conserved. At the beginning and end of the man's walk, we have $L_z = I\dot{\theta}$. It follows that

$$I_0\dot{\theta}_0 = I_1\dot{\theta}_1,$$

where I_0, $\dot{\theta}_0$ are respectively the moment of inertia and angular speed of the man-plus-disc when the man is at the rim, and I_1, $\dot{\theta}_1$ are the corresponding quantities when he is at the centre. Since the man is modelled as a particle, his moment of inertia is mR^2 when at the rim and zero when at the centre. From Example 2, the disc has moment of inertia $\frac{1}{2}MR^2$. Hence we have

$$I_0 = \tfrac{1}{2}MR^2 + mR^2 \quad \text{and} \quad I_1 = \tfrac{1}{2}MR^2.$$

Since also $\dot{\theta}_0 = 1$, the final angular speed of the man-plus-disc is

$$\dot{\theta}_1 = \frac{I_0\dot{\theta}_0}{I_1} = 1 + \frac{2m}{M}.$$

Thus the final angular speed of the disc is $1 + 2m/M$ rad s^{-1} which, as you might have expected, is an increase over the initial value.

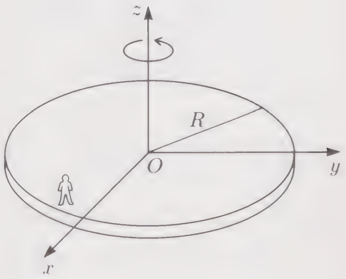


Figure 15

(ii) The initial kinetic energy of the man-plus-disc is $\frac{1}{2}I_0\dot{\theta}_0^2$ and the final kinetic energy is $\frac{1}{2}I_1\dot{\theta}_1^2$. The potential energy remains constant, so the change in total mechanical energy is

$$\tfrac{1}{2}I_1\dot{\theta}_1^2 - \tfrac{1}{2}I_0\dot{\theta}_0^2.$$

Substituting the above values for I_0, I_1, θ_0 and θ_1 into this expression gives an increase in mechanical energy of

$$\tfrac{1}{4}MR^2\left(1 + \frac{2m}{M}\right)^2 - \tfrac{1}{2}\left(\tfrac{1}{2}MR^2 + mR^2\right) = \tfrac{1}{2}mR^2\left(1 + \frac{2m}{M}\right).$$

This mechanical energy can come only from the chemical energy stored in the man's muscles. If he is unable to expend this amount of energy then he cannot reach the centre of the disc. □

Exercise 8

A simple model of a skater consists of three homogeneous cylinders of the same density, as shown in Figure 16. The upright cylinder represents the legs, trunk and head, and has radius 0.12 m and height 1.70 m. The two horizontal cylinders represent the arms, each having radius 0.04 m and length 0.60 m.

The skater spins with negligible friction and air resistance, and the axis of rotation is mid-way between her shoulders. Suppose that she spins initially at a rate ω_0 rad s^{-1} with her arms outstretched horizontally. Estimate the final angular speed when she raises her arms vertically above her shoulders (with the axis of each arm being a distance from the axis of rotation equal to the sum of the radii of the 'arm' and 'body' cylinders).

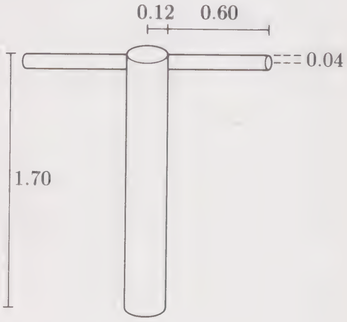


Figure 16

Exercise 9

Any batsperson will tell you that hitting a cricket ball can be quite a painful experience. However, if the ball strikes the bat in just the right place, then the cricketer's hands supply and experience no force during the impact of the ball, and the stroke appears effortless.

Figure 17 shows a simplified model of this situation. The bat is modelled by a thin uniform blade, of mass M and length l. This is assumed to pivot about a *fixed* point O at the top of the blade, which is chosen as the origin of coordinates. The cricket ball is modelled by a particle of mass m, which moves horizontally along the same line before and after hitting the bat, reversing its direction as a result of the impact. Its speeds before and after impact are respectively u and v, and it strikes the bat at a distance b from the pivot. The bat is initially stationary and directed vertically downwards, while immediately after impact its angular speed is $\dot{\theta}$. It is assumed that during the impact no horizontal forces act at O, and that neither gravity nor other external forces supply any torques.

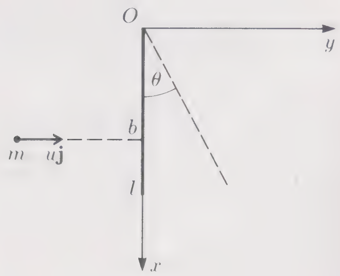


Figure 17

(i) By applying conservation of total linear momentum to the system, show that

$$mu = -mv + \tfrac{1}{2}Ml\dot{\theta}.$$

(ii) By applying conservation of the z-component of total angular momentum to the system, show that

$$mub = -mvb + \tfrac{1}{3}Ml^2\dot{\theta}.$$

(iii) Hence show that the distance b (for painless cricket) is equal to $\frac{2}{3}l$.

[*Solutions on page 49*]

2.4 Applications of the torque law (Audio-tape Subsection)

The problems considered in this subsection involve rigid bodies which rotate about fixed axes while experiencing non-zero torques. In each case, the torque law

$$I\ddot{\theta} = \frac{d}{dt}(I\dot{\theta}) = \dot{L}_z = \Gamma_z^{\text{ext}} \tag{4}$$

applies, though it is now written in the more convenient form

$$I\ddot{\theta} = \Gamma, \tag{10}$$

where $\Gamma = \Gamma_z^{\text{ext}}$.

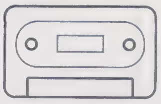
Several of the problems involve the total torque about the origin of the gravitational forces acting on a rigid body. This torque is equal to

$$\mathbf{\Gamma}_{\text{grav}} = \mathbf{R} \times M g\mathbf{k}, \tag{11}$$

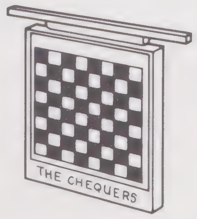
where M is the total mass of the body, $\mathbf{R}$ is the position of its centre of mass and $\mathbf{k}$ is a unit vector vertically downwards.

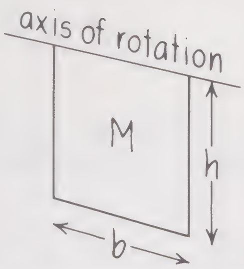
You established this result for any system of particles in Exercise 4 of *Unit 28* Section 4.

Start the audio-tape when you are ready.



1 Example 1: Swinging Pub Sign



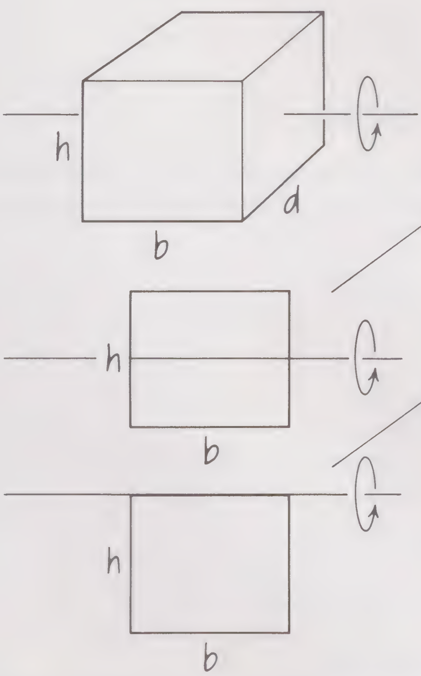


Assumptions:

- Sign is thin uniform rectangular lamina
- Top edge is axis of rotation
- Hinge is frictionless
- Air resistance is negligible

Sign has
mass M, height h, breadth b.

2 Moment of Inertia



$$I_G^{box} = \frac{1}{12}M(h^2 + d^2)$$

For lamina, let $d \to 0$

$$I_G = \boxed{\quad\quad}$$

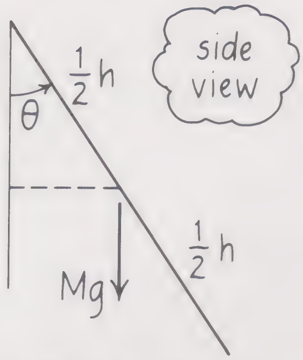
Use parallel axes theorem

$$I = I_G + \boxed{\quad\quad}$$

$$= \boxed{\quad\quad} + \boxed{\quad\quad}$$

$$= \boxed{\quad\quad}$$

3 Torque Law



$\Gamma = \boxed{}$

Torque Law

$\boxed{I\ddot{\theta} = \Gamma}$

$\boxed{}\ \ddot{\theta} = \boxed{}$

$\ddot{\theta} = \boxed{}$

4 Oscillations of the Pub Sign

Equation of motion:

$$\ddot{\theta} + \frac{3g}{2h} \sin\theta = 0$$

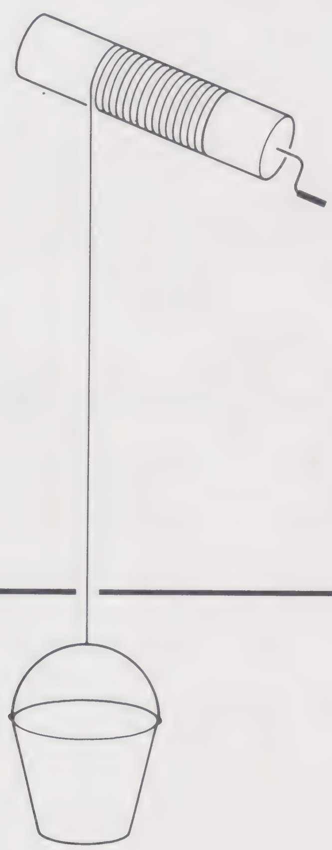
Approximation for <u>small</u> oscillations: $\sin\theta \simeq \theta$

$$\ddot{\theta} + \boxed{} = 0$$

Angular frequency: $\omega = \boxed{}$

Period: $T = \boxed{}$

5 Example 2: Water Well



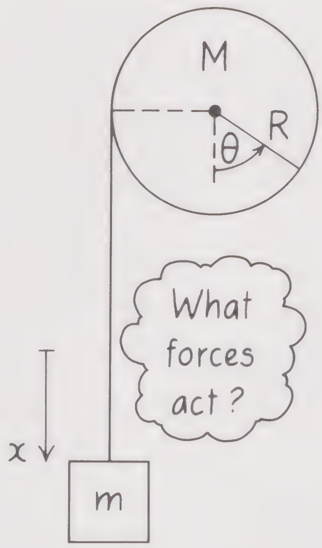
Assumptions:

- Drum is uniform cylinder
- Axle is horizontal and frictionless
- Rope is light and inextensible
- Bucket is modelled by a particle
- Air resistance is negligible

Drum has mass M, radius R.

Bucket has mass m.

6 Equations of Motion



Bucket

$F = \boxed{}$ ← x-component of total force

$a = \boxed{}$ ← in terms of x

$\boxed{} = \boxed{}$ ← $ma = F$

Drum

$I = \boxed{}$; $\Gamma = \boxed{}$

$\boxed{} = \boxed{}$ ← $I\ddot{\theta} = \Gamma$

7 Connection Between x and θ

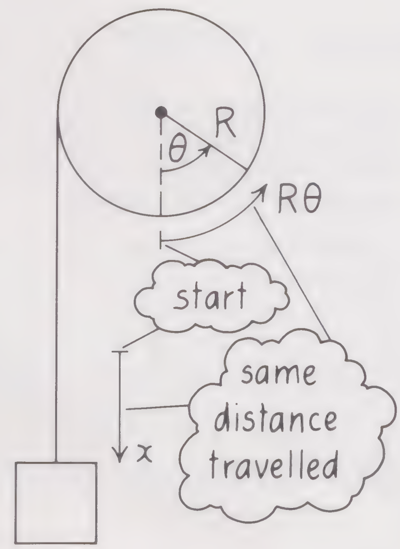
Equations of motion:

Bucket: $m\ddot{x} = mg - T$

Drum: $\frac{1}{2}MR\ddot{\theta} = T$

Adding to eliminate T:

$$\boxed{m\ddot{x} + \frac{1}{2}MR\ddot{\theta} = mg}$$



Geometrical connection between coordinates:

$$\boxed{x = R\theta}$$

Eliminate θ:

$$m\ddot{x} + \boxed{} = mg$$

$$\left(\boxed{}\right)\ddot{x} = mg$$

$$\ddot{x} = \frac{mg}{\boxed{}} = \frac{g}{\boxed{}}$$

Check limiting case $M = 0$

The following exercises provide further practice in the application of the torque law (Equation (10)) to rigid bodies, when the axis of rotation is fixed.

Exercise 10

A pendulum consists of a light rod of length l which is attached to the surface of a homogeneous sphere of radius R and mass M. The rod is attached to a fixed pivot and swings in a vertical plane. Assuming that friction and air resistance may be ignored, and that the oscillations are small, use the torque law to find the period of the pendulum in terms of l and R.

Exercise 11

Solve the problem described in Exercise 10 by applying conservation of mechanical energy.

Exercise 12

A door of mass M and breadth b is initially closed. It is opened by a force applied at the handle, which is mid-way down one edge. This force has constant magnitude P and is always at right angles to the plane of the door. What is the speed of the handle when the door has swung through an angle of $\frac{1}{2}\pi$ rad?

Exercise 13

A compound pendulum consists of a plane lamina of mass M, suspended from a fixed point O which is a distance l from its centre of mass G. The lamina has moment of inertia I_G about an axis which is perpendicular to its plane and passes through G. Show that the period of small oscillations in the plane of the lamina is given by

$$\tau = 2\pi\sqrt{\frac{I_G + Ml^2}{Mgl}}.$$

[*Solutions on page 50*]

Summary of Section 2

1. A **rigid body** is a many-particle system with the property that all the inter-particle distances remain constant in time. The inter-particle forces in a rigid body are assumed to satisfy the strong form of Newton's third law, so that the torque law $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ applies.

2. **Rigid-body rotation** about a fixed axis is a rigid-body motion for which particles on the axis remain fixed, while other particles move in circles centred on and perpendicular to the axis. If the axis of rotation is chosen to be the z-axis, then all particles have the same angular velocity $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$. If $\mathbf{r}_i$ is the position of Particle i, then its velocity is given by

$$\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \mathbf{r}_i = \dot{\theta}\mathbf{k} \times \mathbf{r}_i.$$

3. The z-component of the total angular momentum $\mathbf{L}$ of the body described above is

$$L_z = I\dot{\theta},$$

where

$$I = \sum_{i=1}^{n} m_i d_i^2$$

is the **moment of inertia** of the body about the axis of rotation (the sum being taken over all particles of the body). It follows from the torque law that

$$I\ddot{\theta} = \dot{L}_z = \Gamma_z^{\text{ext}}.$$

The kinetic energy of the body is given by

$$T = \tfrac{1}{2}I\dot{\theta}^2.$$

4. The moment of inertia of a continuous rigid body about a given axis of rotation is

$$I = \int_B \rho(\mathbf{r})d^2(\mathbf{r})\,dV,$$

where $\rho(\mathbf{r})$ is the density of mass at point $\mathbf{r}$, $d(\mathbf{r})$ is the distance of the point $\mathbf{r}$ from the axis of rotation, and the integration is over the volume of the body B.

For a homogeneous body, the density ρ is constant. The moments of inertia of certain homogeneous rigid bodies about axes through their centres of mass are given in Table 1 on pages 21 and 22.

5. The **parallel axes theorem** states that the moment of inertia of a rigid body about an axis A can be written as

$$I = I_G + Md^2,$$

where M is the mass of the body, d is the distance of the centre of mass G from the axis A, and I_G is the moment of inertia of the body about an axis B which passes through G and is parallel to A.

3 Rotation about a moving axis with fixed orientation

This section extends the ideas of Section 2 to objects that move bodily through space whilst spinning about an axis of rotation. However, to keep the discussion as simple as possible, we shall suppose that the axis of rotation maintains a fixed orientation.

Figure 1 below shows an example of the type of motion to be considered, namely, a uniform cylinder rolling down a slope. Here, the axis of rotation passes through the centre of the cylinder and points in a horizontal direction, parallel to the z-axis. You have seen motion similar to this in the television programme for *Unit 27*. In Subsections 3.2 and 3.3 you are asked to make some predictions about it, but first it is necessary to discuss some useful theorems.

3.1 Centre of mass decomposition theorems

Figure 1 shows a system of n particles which form a cylinder, but for the moment we shall ignore the particular form of this system and consider what can be said about a system whose configuration is arbitrary.

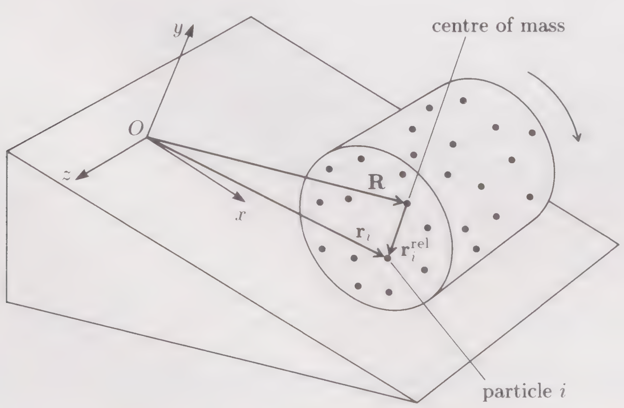


Figure 1

The diagram indicates the position $\mathbf{R}$ of the centre of mass, and the position $\mathbf{r}_i$ of the ith particle (both relative to the origin of a static Cartesian coordinate system). The notation $\mathbf{r}_i^{\text{rel}}$ describes the displacement vector from the centre of mass to the ith particle. From the definition of vector addition, we have

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}_i^{\text{rel}}, \tag{1}$$

so that the position vector of the ith particle can be regarded as the vector sum of the position vector of the centre of mass and the displacement vector from the centre of mass to the ith particle. On differentiating Equation (1), we obtain a similar result for the velocity of the ith particle, that is,

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{r}}_i^{\text{rel}}. \tag{2}$$

In terms of these quantities, we can state the following theorems.

Centre of mass decomposition theorems

Consider any system of n particles. As usual, let m_i, $\mathbf{r}_i$ and $\dot{\mathbf{r}}_i$ denote the mass, position and velocity of the ith particle, and let $\mathbf{F}_i^{\text{ext}}$ be the external force acting on it. Let M, $\mathbf{R}$ and $\dot{\mathbf{R}}$ denote the total mass and the position and velocity of the centre of mass of the system, and let $\mathbf{F}^{\text{ext}}$ be the total external force acting on it.

Then the total kinetic energy T, total angular momentum $\mathbf{L}$ and total external torque $\mathbf{\Gamma}^{\text{ext}}$ (both about the origin) can be decomposed as follows:

$$T = T_G + \tfrac{1}{2}M|\dot{\mathbf{R}}|^2, \tag{3}$$

$$\mathbf{L} = \mathbf{L}_G + \mathbf{R} \times M\dot{\mathbf{R}}, \tag{4}$$

$$\mathbf{\Gamma}^{\text{ext}} = \mathbf{\Gamma}_G + \mathbf{R} \times \mathbf{F}^{\text{ext}}, \tag{5}$$

where

$$T_G = \tfrac{1}{2}\sum_{i=1}^{n} m_i|\dot{\mathbf{r}}_i^{\text{rel}}|^2, \quad \mathbf{L}_G = \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times m_i\dot{\mathbf{r}}_i^{\text{rel}}, \quad \mathbf{\Gamma}_G = \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times \mathbf{F}_i^{\text{ext}}.$$

In spite of their complicated appearance, Equations (3)–(5) have a straightforward interpretation. For example, the term

$$\tfrac{1}{2}M|\dot{\mathbf{R}}|^2$$

on the right-hand side of Equation (3) is the kinetic energy of a single particle of mass M situated at the centre of mass. It is referred to as the **kinetic energy of the centre of mass**. The first term on the right-hand side is

$$T_G = \tfrac{1}{2}\sum_{i=1}^{n} m_i|\dot{\mathbf{r}}_i^{\text{rel}}|^2.$$

This is the kinetic energy that would be perceived by someone stationed at the centre of mass, and is referred to as the **kinetic energy relative to the centre of mass**.

Equation (3) states then that the total kinetic energy can be split up into two terms, of which one is associated with motion *of* the centre of mass and the other is associated with motion *relative* to the centre of mass. A similar interpretation applies to Equations (4) and (5). We have emphasized this decomposition by using the subscript G for all quantities defined *relative* to the centre of mass. A further 'centre of mass decomposition' result is the parallel axes theorem for moments of inertia,

$$I = I_G + Md^2,$$

which was stated and proved in Subsection 2.2.

Proofs of the centre of mass decomposition theorems

If you do not wish to consider the proofs of these theorems, then you may like to move directly to Exercise 1 below.

To establish Equation (3), we start by noting that the total kinetic energy is defined by

$$T = \sum_{i=1}^{n} \tfrac{1}{2}m_i|\dot{\mathbf{r}}_i|^2.$$

Using Equation (2), the expression for T_G may be rewritten as

$$\begin{aligned} T_G = \sum_{i=1}^{n} \tfrac{1}{2}m_i|\dot{\mathbf{r}}_i^{\text{rel}}|^2 &= \tfrac{1}{2}\sum_{i=1}^{n} m_i|\dot{\mathbf{r}}_i - \dot{\mathbf{R}}|^2 \\ &= \tfrac{1}{2}\sum_{i=1}^{n} m_i|\dot{\mathbf{r}}_i|^2 - \sum_{i=1}^{n} m_i\dot{\mathbf{r}}_i \cdot \dot{\mathbf{R}} + \tfrac{1}{2}\sum_{i=1}^{n} m_i|\dot{\mathbf{R}}|^2 \\ &= T - \left(\sum_{i=1}^{n} m_i\dot{\mathbf{r}}_i\right) \cdot \dot{\mathbf{R}} + \tfrac{1}{2}|\dot{\mathbf{R}}|^2\sum_{i=1}^{n} m_i. \end{aligned}$$

Recalling that

$$M = \sum_{i=1}^{n} m_i, \quad M\mathbf{R} = \sum_{i=1}^{n} m_i \mathbf{r}_i \quad \text{and} \quad M\dot{\mathbf{R}} = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i,$$

by the definitions of total mass and centre of mass, we have

$$T_G = T - M|\dot{\mathbf{R}}|^2 + \tfrac{1}{2}M|\dot{\mathbf{R}}|^2 = T - \tfrac{1}{2}M|\dot{\mathbf{R}}|^2,$$

leading immediately to Equation (3).

Equation (4) can be established in a similar way. The total angular momentum about the origin is defined by

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i.$$

Expansion of the expression for $\mathbf{L}_G$ gives

$$\begin{aligned}
\mathbf{L}_G &= \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times m_i \dot{\mathbf{r}}_i^{\text{rel}} \\
&= \sum_{i=1}^{n} [(\mathbf{r}_i - \mathbf{R}) \times m_i(\dot{\mathbf{r}}_i - \dot{\mathbf{R}})] \\
&= \sum_{i=1}^{n} \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i - \sum_{i=1}^{n} \mathbf{r}_i \times m_i \dot{\mathbf{R}} - \sum_{i=1}^{n} \mathbf{R} \times m_i \dot{\mathbf{r}}_i + \sum_{i=1}^{n} \mathbf{R} \times m_i \dot{\mathbf{R}} \\
&= \mathbf{L} - \left(\sum_{i=1}^{n} m_i \mathbf{r}_i\right) \times \dot{\mathbf{R}} - \mathbf{R} \times \left(\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i\right) + \mathbf{R} \times \left(\sum_{i=1}^{n} m_i\right) \dot{\mathbf{R}} \\
&= \mathbf{L} - M\mathbf{R} \times \dot{\mathbf{R}} - \mathbf{R} \times M\dot{\mathbf{R}} + \mathbf{R} \times M\dot{\mathbf{R}} \\
&= \mathbf{L} - \mathbf{R} \times M\dot{\mathbf{R}},
\end{aligned}$$

as required. Equation (5) is simpler to prove, since

$$\boldsymbol{\Gamma}^{\text{ext}} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}$$

and

$$\begin{aligned}
\boldsymbol{\Gamma}_G = \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times \mathbf{F}_i^{\text{ext}} &= \sum_{i=1}^{n} (\mathbf{r}_i - \mathbf{R}) \times \mathbf{F}_i^{\text{ext}} \\
&= \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} - \mathbf{R} \times \sum_{i=1}^{n} \mathbf{F}_i^{\text{ext}} \\
&= \boldsymbol{\Gamma}^{\text{ext}} - \mathbf{R} \times \mathbf{F}^{\text{ext}}.
\end{aligned}$$

End of proofs

Exercise 1

Show that, for a system of particles acted upon only by gravity, the decomposition theorem for total external torque (Equation (5)) reduces to $\boldsymbol{\Gamma}^{\text{ext}} = \mathbf{R} \times \mathbf{F}^{\text{ext}}$.

You showed essentially this result in Exercise 4 of *Unit 28* Section 4.

Exercise 2

A system of particles is acted upon by a central force (that is, a force directed towards or away from the origin) so that the force on the ith particle is of the form $\mathbf{F}_i^{\text{ext}} = \lambda \mathbf{r}_i$, where λ may depend on $\mathbf{r}_i$. Show that

$$\boldsymbol{\Gamma}_G = -\mathbf{R} \times \mathbf{F}^{\text{ext}}.$$

Exercise 3

A vertical uniform disc of mass M and radius R has two particles of mass m attached to diametrically opposite points of its rim, and rolls along a horizontal plane with speed v (that is, its centre has speed v). Since the disc rolls without slipping, its angular speed about the centre is equal to v/R (as you will see shown in the next subsection). Use the centre of mass decomposition theorem (Equation (3)) to show that the kinetic energy of this system is $(\frac{3}{4}M + 2m)v^2$.

[*Solutions on page 51*]

3.2 Using energy conservation for rolling objects

The centre of mass decomposition theorems allow us to predict the motion of rolling objects, by using either energy conservation or the torque law. In this subsection the law of conservation of total mechanical energy is applied, whereas in the next subsection the use of the torque law is illustrated.

Example 1

A homogeneous solid cylinder of mass M and radius R rolls without slipping down a rough plane which is inclined at an angle α to the horizontal. The total mechanical energy of the cylinder is conserved. What is the magnitude of the acceleration of its centre of mass?

Solution

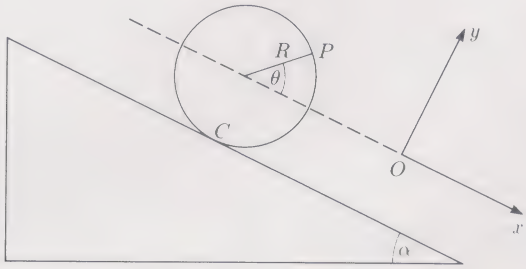


Figure 2

Figure 2 shows the cylinder in profile and the chosen coordinate system. The x-axis points directly down the slope of the plane, and the z-axis (which points out of the page) is parallel to the axis of rotation of the cylinder. If the cylinder is modelled as a system of particles, then we know from Equation (3) that its total kinetic energy can be written as

$$T = \tfrac{1}{2}M|\dot{\mathbf{R}}|^2 + T_G,$$

where T_G is the kinetic energy relative to the centre of mass. Since the centre of mass moves along the x-axis, with $\mathbf{R} = X\mathbf{i}$, the equation for T becomes

$$T = \tfrac{1}{2}M\dot{X}^2 + T_G.$$

This expression can be further elaborated because the cylinder is a rigid body which moves in a special way (it rolls). The rigidity of the cylinder means that its motion relative to the axis of rotation is a rigid-body rotation. From Equation (5) of Section 2, it follows that the kinetic energy relative to the centre of mass is $T_G = \frac{1}{2}I\dot{\theta}^2$, where the moment of inertia about the axis of rotation is (from Table 1 on page 21) $I = \frac{1}{2}MR^2$. The total kinetic energy is now given by

$$T = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{4}MR^2\dot{\theta}^2.$$

Now the fact that the cylinder rolls, and does not slip, implies a relationship between $\dot{X}$ and $\dot{\theta}$. The rate of increase of X is equal to the rate at which the cylinder 'rolls out' its circumference which, in turn, is given by the radius of the cylinder times the rate of decrease of θ (see Figure 2). Thus

$$\dot{X} = R(-\dot{\theta}).$$

[As an alternative and less intuitive justification of this 'rolling condition', the position vector $\mathbf{r}$ of a particle P on the surface of the cylinder is given by (see Figure 2)

$$\mathbf{r} = X\mathbf{i} + R\cos\theta\,\mathbf{i} + R\sin\theta\,\mathbf{j},$$

whence its velocity is

$$\dot{\mathbf{r}} = \dot{X}\mathbf{i} - R\dot{\theta}\sin\theta\,\mathbf{i} + R\dot{\theta}\cos\theta\,\mathbf{j}.$$

The point C of the cylinder, which has instantaneous contact with the plane, has velocity zero, and $\theta = -\frac{1}{2}\pi$ at this point. Thus $\mathbf{0} = (\dot{X} + R\dot{\theta})\mathbf{i}$, giving $\dot{X} = -R\dot{\theta}$ as before.]

The total kinetic energy is therefore

$$T = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{4}M\dot{X}^2 = \tfrac{3}{4}M\dot{X}^2. \qquad (6)$$

The potential energy U of the cylinder can be calculated by adding together the gravitational potential energies of all of its particles. Since the x-axis points directly down the slope of the plane, we obtain (taking the origin as datum)

$$U = \sum_{i=1}^{n} m_i g(-x_i \sin\alpha) = -\left(\sum_{i=1}^{n} m_i x_i\right) g\sin\alpha = -MXg\sin\alpha. \qquad (7)$$

Note that this is the gravitational potential energy of the 'representative particle' of mass M placed at the centre of mass.

By Equations (6) and (7), the total mechanical energy of the cylinder is

$$E = \tfrac{3}{4}M\dot{X}^2 - MXg\sin\alpha.$$

Since E is constant, by the law of conservation of total mechanical energy, differentiation gives

$$\tfrac{3}{2}M\dot{X}\ddot{X} - Mg\dot{X}\sin\alpha = 0.$$

The cylinder is not stationary (and has non-zero mass), so we can divide the last equation by $\tfrac{3}{2}M\dot{X}$ to obtain

$$\ddot{X} = \tfrac{2}{3}g\sin\alpha.$$

This is the required magnitude of the acceleration of the cylinder's centre of mass. □

Before proceeding to comment upon the result of this example, we highlight an important result which was derived as a by-product of the solution.

Rolling condition

If a cylinder or sphere of radius R rolls (without slipping) along a rough plane, then the speed v of the centre is related to the angular speed $|\dot{\theta}|$ of the body about its centre by the equation

$$v = R|\dot{\theta}|.$$

We have three comments to make about the result of Example 1.

First, notice that the acceleration of the centre of mass is *less* than the acceleration $g\sin\alpha$ which would arise if the cylinder slid down the plane with no friction. One way of understanding this is to consider how the quantity on the right-hand side of Equation (6) is made up. The centre of mass motion has kinetic energy $\tfrac{1}{2}M\dot{X}^2$, while the rotation has an associated kinetic energy $\tfrac{1}{4}M\dot{X}^2$. Thus one third of the kinetic energy of the cylinder is 'locked up' in rotation, leaving two thirds for the motion of the centre of mass. As the cylinder descends the slope it loses potential energy, but only two thirds of this energy is converted into the kinetic energy of the centre of mass. The same factor of two thirds occurs in the acceleration of the centre of mass, because the acceleration of a particle is proportional to the gradient of its potential energy.

Next, consider the x-component F_x of the total external force $\mathbf{F}^{\text{ext}}$ acting on the cylinder. You saw in *Unit 17* that this component is equal to $M\ddot{X}$, so the result of Example 1 gives

Unit 17 Subsection 2.3

$$F_x = \tfrac{2}{3}Mg\sin\alpha.$$

This is less than the component, $Mg\sin\alpha$, of the gravitational force down the slope of the plane. The reason for this difference is that the cylinder experiences a force due to friction which opposes its motion. Denoting the magnitude of the frictional force by S, the x-component of the total external force is

$$F_x = Mg\sin\alpha - S.$$

Thus the magnitude of the frictional force is

$$S = Mg\sin\alpha - F_x = \tfrac{1}{3}Mg\sin\alpha.$$

If there were no friction, then the cylinder would slide rather than roll.

The presence of this force is essential in order for the cylinder to roll without slipping.

Finally, you might question the use of conservation of mechanical energy in a situation where friction plays an important role. In fact, mechanical energy is conserved provided that the cylinder remains rigid and rolls without slipping. This is because the part of a rolling cylinder which is in contact with the ground is momentarily at rest. Thus the frictional force acts on stationary particles and therefore has no effect on their energy, any more than it would if the cylinder were static.

Exercise 4

The disc described in Exercise 3 rolls without slipping down a plane which is inclined at an angle α to the horizontal. Use the method of Example 1 to show that the acceleration of its centre of mass is $(M + 2m)g \sin\alpha/(\frac{3}{2}M + 4m)$.

Exercise 5

A steamroller consists of two solid uniform rollers of radii R_1, R_2 and masses M_1, M_2, and a cabin of mass M. Neglecting internal friction, find the magnitude of the steamroller's acceleration when it rolls down a hill which slopes at an angle α to the horizontal.

[*Solutions on page 51*]

3.3 Using the torque law for rolling objects

An alternative way of predicting the motion of rolling objects is based on the torque law for a system of particles,

Equation (8) of Section 1

$$\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}.$$

Combining this with the centre of mass decomposition theorems for angular momentum and for torque (Equations (4) and (5)) gives

$$\dot{\mathbf{L}}_G + \frac{d}{dt}(\mathbf{R} \times M\dot{\mathbf{R}}) = \mathbf{\Gamma}_G + \mathbf{R} \times \mathbf{F}^{\text{ext}}. \tag{8}$$

However, we have

$$\begin{aligned}\frac{d}{dt}(\mathbf{R} \times M\dot{\mathbf{R}}) &= \mathbf{R} \times M\ddot{\mathbf{R}} \qquad (\text{since } \dot{\mathbf{R}} \times M\dot{\mathbf{R}} = \mathbf{0})\\ &= \mathbf{R} \times \mathbf{F}^{\text{ext}} \qquad (\text{since } M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}} \text{ from } \textit{Unit 17}).\end{aligned}$$

Unit 17 Subsection 2.3

Equation (8) therefore leads directly to the following result.

Torque law relative to the centre of mass

For any system of particles, the rate of change of the total angular momentum relative to the centre of mass is equal to the total external torque relative to the centre of mass, that is,

$$\dot{\mathbf{L}}_G = \mathbf{\Gamma}_G. \tag{9}$$

For a rigid body whose axis of rotation passes through the centre of mass and remains parallel to the z-axis, the z-component of Equation (9) leads to the equation

$$I\ddot{\theta} = \Gamma_G, \tag{10}$$

This is the counterpart of Equation (10) in Section 2 for the case of a moving axis with fixed orientation.

where I is the moment of inertia about the axis of rotation, $\dot{\theta}\mathbf{k}$ is the angular velocity of the body relative to this axis, and $\Gamma_G = \mathbf{\Gamma}_G \cdot \mathbf{k}$. The following example illustrates how this result may be put to use.

Example 2

Figure 3 shows a solid sphere, of radius R and mass M, rolling down a plane which is inclined at an angle α to the horizontal. The diagram indicates the three forces acting on the sphere: gravity, of magnitude Mg, a normal reaction of magnitude N, and a frictional reaction of magnitude S. The x-axis is directed down the slope.

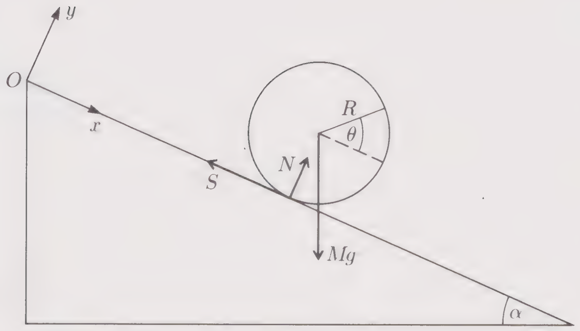


Figure 3

(i) By considering the motion of the sphere's centre of mass, show that

$$M\ddot{X} = Mg\sin\alpha - S \quad \text{and} \quad N = Mg\cos\alpha.$$

(ii) Use the torque law relative to the centre of mass (in the form of Equation (10)) to show that

$$S = \tfrac{2}{5}M\ddot{X}.$$

(iii) Hence show that the sphere cannot simply roll (but will also slip) if

$$\tan\alpha > \tfrac{7}{2}\mu,$$

where μ is the coefficient of static friction.

Solution

(i) The total external force acting on the sphere is

$$\begin{aligned}\mathbf{F}^{\text{ext}} &= -S\mathbf{i} + N\mathbf{j} + Mg(\sin\alpha\,\mathbf{i} - \cos\alpha\,\mathbf{j})\\ &= (Mg\sin\alpha - S)\mathbf{i} + (N - Mg\cos\alpha)\mathbf{j}.\end{aligned}$$

The centre of mass of the sphere moves in accordance with the equation

$$M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}},$$

so that on taking x- and y-components, we have

$$M\ddot{X} = Mg\sin\alpha - S \tag{11}$$

and (since $\ddot{Y} = 0$)

$$N = Mg\cos\alpha. \tag{12}$$

(ii) The total external torque relative to the centre of mass of the sphere is

$$\mathbf{\Gamma}_G = (-R\mathbf{j}\times N\mathbf{j}) + (-R\mathbf{j}\times(-S\mathbf{i})) = -RS\mathbf{k},$$

so that from Equation (10) we obtain

$$I\ddot{\theta} = -RS. \tag{13}$$

I is the moment of inertia of the sphere about an axis through its centre of mass.

The condition for rolling without slipping is $\dot{X} = -R\dot{\theta}$, which after differentiation and rearrangement gives $\ddot{\theta} = -\ddot{X}/R$. Substitution for $\ddot{\theta}$ in Equation (13) provides

$$S = \frac{I}{R^2}\ddot{X} = \beta M\ddot{X}, \tag{14}$$

The coefficient β is introduced here and used below because the analysis applies also to rolling bodies of circular cross-section other than a solid sphere. This analysis is relevant to Exercises 6 and 7 below.

where $\beta = I/(MR^2)$. For a solid sphere, Table 1 of Section 2 (page 21) gives $\beta = \frac{2}{5}$, so that

$$S = \tfrac{2}{5}M\ddot{X}.$$

(iii) Combining Equations (11) and (14) produces

$$M\ddot{X} = Mg\sin\alpha - \beta M\ddot{X}$$

or $$\ddot{X} = \frac{g\sin\alpha}{1+\beta}.$$

Note that this result agrees with that of Example 1, where $\beta = \frac{1}{2}$ and
$$\ddot{X} = \tfrac{2}{3}g\sin\alpha.$$

From Equation (14), we then have

$$S = \frac{\beta Mg\sin\alpha}{1+\beta},$$

which on applying Equation (12) becomes

$$\frac{S}{N} = \left(\frac{\beta}{1+\beta}\right)\tan\alpha.$$

This must be less than or equal to the coefficient of static friction μ, so that

$$\tan\alpha \le \left(\frac{1+\beta}{\beta}\right)\mu.$$

The coefficient of static friction is more appropriate here than the coefficient of kinetic friction, because the point of contact of the sphere with the inclined plane is momentarily at rest.

For a solid sphere, we have $\beta = \frac{2}{5}$, so

$$\tan\alpha \le \tfrac{7}{2}\mu.$$

If $\tan\alpha > \frac{7}{2}\mu$, then the sphere will slip to some extent and will not undergo a pure rolling motion. □

Exercise 6

A thin-walled hollow cylinder rolls without slipping down a plane which is at an angle $\frac{1}{4}\pi$ to the horizontal.

(i) By modifying the solution to Example 1, predict the magnitude of the cylinder's acceleration.

(ii) Hence show, as in Example 2, that the cylinder cannot just roll (but will also slide) if the coefficient of static friction between the cylinder and the plane is less than $\frac{1}{2}$.

Exercise 7

Two spheres of identical mass and radius are released from rest on an inclined plane, and roll down the plane without slipping for the same length of time. One of the spheres is solid, and the other is a hollow shell of negligible thickness. Show that the ratio of the distances travelled by the solid and hollow spheres is 25:21.

Exercise 8

Figure 4 shows a spanner which has been thrown in such a way that it tumbles head-over-heels, rotating about a horizontal axis which maintains a fixed orientation perpendicular to the page. Neglecting air resistance, and using the torque law relative to the centre of mass, show that the angle θ between the spanner and the horizontal varies at a constant rate.

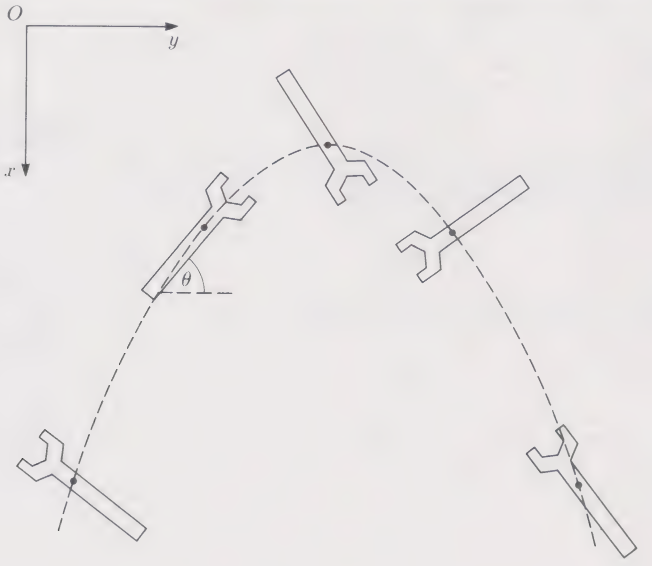


Figure 4

Exercise 9

Figure 5 shows a skater of mass M who moves at constant speed v around a horizontal circle of radius R. She skates with her feet together, and leans inwards towards the centre of the circle. The line from her centre of mass to her skates makes an angle α with the vertical.

(i) Find the total external torque $\mathbf{\Gamma}_G$ relative to her centre of mass. (You may use the fact, from *Unit 28* Subsection 3.1, that an object travelling at uniform speed v around a circle of radius R experiences an acceleration of magnitude v^2/R directed towards the centre of the circle.)

(ii) As an approximation, it is assumed that all particles within the skater have the same velocity at each instant. By using the torque law relative to the centre of mass, show that this approximation leads to the conclusion that

$$\tan\alpha = \frac{v^2}{gR}.$$

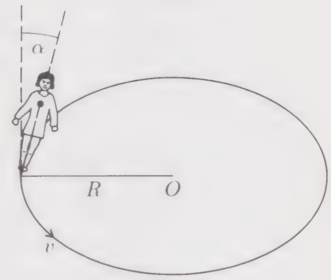


Figure 5

[*Solutions on page 52*]

Summary of Section 3

1. Consider a system of n particles, with centre of mass $\mathbf{R}$, total mass M, total kinetic energy T and total angular momentum $\mathbf{L}$ about the origin, which is acted upon by a total external force $\mathbf{F}^{\text{ext}}$ corresponding to a total external torque $\mathbf{\Gamma}^{\text{ext}}$ about the origin. Suppose that the ith particle has mass m_i and position $\mathbf{r}_i$, and experiences an external force $\mathbf{F}_i^{\text{ext}}$. Then the position of the particle relative to the centre of mass, $\mathbf{r}_i^{\text{rel}}$, is defined by

$$\mathbf{r}_i^{\text{rel}} = \mathbf{r}_i - \mathbf{R}.$$

The **centre of mass decomposition theorems** are as follows.

(i) The total kinetic energy of the system is given by

$$T = T_G + \tfrac{1}{2}M|\dot{\mathbf{R}}|^2,$$

where

$$T_G = \tfrac{1}{2}\sum_{i=1}^{n} m_i|\dot{\mathbf{r}}_i^{\text{rel}}|^2$$

is the kinetic energy relative to the centre of mass.

(ii) The total angular momentum of the system about the origin is given by

$$\mathbf{L} = \mathbf{L}_G + \mathbf{R} \times M\dot{\mathbf{R}},$$

where

$$\mathbf{L}_G = \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times m_i\dot{\mathbf{r}}_i^{\text{rel}}$$

is the angular momentum about the centre of mass.

(iii) The total external torque of the system about the origin is given by

$$\mathbf{\Gamma}^{\text{ext}} = \mathbf{\Gamma}_G + \mathbf{R} \times \mathbf{F}^{\text{ext}},$$

where

$$\mathbf{\Gamma}_G = \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times \mathbf{F}_i^{\text{ext}}$$

is the total external torque about the centre of mass.

2. For a rigid body rotating with angular velocity $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$ about an axis which passes through the centre of mass and is always aligned with the z-axis, the kinetic energy and the z-component of the total angular momentum relative to the centre of mass are given respectively by

$$T_G = \tfrac{1}{2}I\dot{\theta}^2 \quad \text{and} \quad \mathbf{L}_G \cdot \mathbf{k} = I\dot{\theta},$$

where I is the moment of inertia of the body about the axis of rotation.

3. For any system of particles, the rate of change of the total angular momentum relative to the centre of mass is equal to the total external torque relative to the centre of mass, that is,

$$\dot{\mathbf{L}}_G = \mathbf{\Gamma}_G.$$

This is the **torque law relative to the centre of mass**.

4. If a cylinder or sphere of radius R rolls (without slipping) along a rough plane, then the speed v of the centre is related to the angular speed $|\dot{\theta}|$ of the body about its centre by the **rolling condition**

$$v = R|\dot{\theta}|.$$

4 Rotation about an axis whose orientation changes

In this section the previous theory of the unit is extended one stage further, to rigid bodies which rotate about axes whose orientations change. This extension provides the answers to some interesting questions. Why does a spinning top balance on its point? Why does a gyroscope move sideways when a downward force is applied to it? In the television programme (Subsection 4.1) you will see these phenomena first demonstrated and then explained. In Subsection 4.2, the rate of precession of the spin axis of a top is expressed in terms of its angular speed.

4.1 Tops, gyroscopes and angular momentum (Television Subsection)

The television programme uses some notation different to that employed elsewhere in the unit, as follows.

1. The symbol $\boldsymbol{l}_Q$ is used (instead of $\boldsymbol{l}$) to denote the angular momentum of a single particle relative to the point Q, and $(\boldsymbol{l}_i)_Q$ replaces $\boldsymbol{l}_i$ for the angular momentum of the ith particle within a system of particles. Similarly, the symbol $\mathbf{L}_Q$ (rather than $\mathbf{L}$) denotes the total angular momentum of the system relative to the point Q, and $\mathbf{\Gamma}_Q$ (instead of $\mathbf{\Gamma}^{\text{ext}}$) is the total external torque relative to Q.

The subscript Q denotes that the point Q has been taken as the origin.

2. The notation $\mathbf{F}_i^{\text{E}}$ is used (instead of $\mathbf{F}_i^{\text{ext}}$) for the external force on the ith particle.
3. The position of the centre of mass is denoted by $\mathbf{r}$, rather than by $\mathbf{R}$.

Now watch the television programme.
Read the following after viewing the programme.

TV29

The television programme consisted of the following elements:

(i) a brief review of the theory of rotating objects;

(ii) demonstrations of the behaviour of spinning tops and gyroscopes, and an explanation for these phenomena arising from the torque law;

(iii) a description of the application of a gyroscope in controlling the rolling motion of a ship.

The programme began with a motivational demonstration that a spinning top does not fall over, even when its axis of rotation leans away from the vertical. Paul Clark then reviewed the definitions of angular momentum for an individual particle and for a system of particles.

See Equations (1) and (6) of Section 1.

Allan Solomon applied these definitions to a 'top' formed from a bicycle wheel which spins about a vertical shaft passing through the centre of the wheel and supported at its lower end. Part of the model used to represent the wheel is shown in Figure 1. The origin O of the coordinate system is on the axis of rotation, and the vertical shaft lies along the z-axis. The diagram indicates the positions of two diametrically-opposed identical particles on the rim of the wheel. At the instant shown, the velocity of

Particle 1 points directly into the page, and the velocity of Particle 2 points directly out of the page. The masses, speeds and distances from the origin of the two particles are the same.

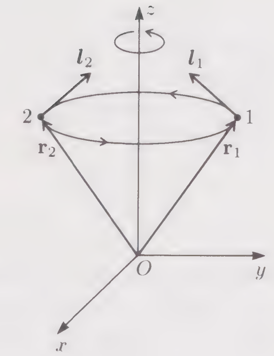


Figure 1

Then, using the right hand rule to determine the directions of the individual angular momentum vectors

$$\boldsymbol{l}_1 = \mathbf{r}_1 \times m_1\mathbf{v}_1 \quad \text{and} \quad \boldsymbol{l}_2 = \mathbf{r}_2 \times m_2\mathbf{v}_2,$$

it can be seen that these vectors lie in the plane of the paper in the directions shown. Moreover, since the particles are identical in terms of mass, speed and distance from the origin, we have

$$|\boldsymbol{l}_1| = m_1|\mathbf{r}_1||\mathbf{v}_1| = m_2|\mathbf{r}_2||\mathbf{v}_2| = |\boldsymbol{l}_2|,$$

showing that the magnitudes of $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are the same. Thus, using the parallelogram law for vector addition, the sum of these angular momenta lies along the axis of rotation. A similar result applies to other pairs of diametrically-opposed particles constituting the wheel, whose symmetry implies that it is wholly composed of such particle pairs. The conclusion drawn is that the total angular momentum of the wheel is along the fixed axis of rotation.

The parallelogram law is stated in *Unit 14* Subsection 1.4.

The same approach can be applied to any rigid body rotating about a fixed axis of symmetry. An alternative argument, not given in the programme, is as follows. According to a result in Subsection 2.1 (page 16), the total angular momentum of a rigid body which rotates about the z-axis is

$$\mathbf{L} = \dot{\theta}\sum_{i=1}^{n} m_i(-x_iz_i\mathbf{i} - y_iz_i\mathbf{j} + (x_i^2 + y_i^2)\mathbf{k}). \tag{1}$$

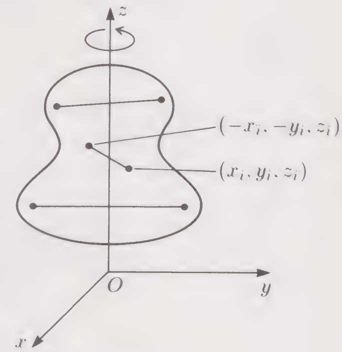


Figure 2

In general, this vector has non-zero **i**- and **j**-components, and so does not have the same direction as $\pm\mathbf{k}$. This means that the total angular momentum vector does not, in general, point along the fixed axis of rotation. However, if the body is symmetric about the rotation axis, then Equation (1) can be greatly simplified. Taking the z-axis to be the axis of rotation once more, Figure 2 shows that in this symmetric case each particle in the body with coordinates (x_i, y_i, z_i) has an identical counterpart on the far side of the z-axis with coordinates $(-x_i, -y_i, z_i)$. Hence the sums

$$\sum_{i=1}^{n} m_ix_iz_i \quad \text{and} \quad \sum_{i=1}^{n} m_iy_iz_i$$

are both zero, and Equation (1) reduces to

$$\mathbf{L} = \dot{\theta}\sum_{i=1}^{n} m_i(x_i^2 + y_i^2)\mathbf{k} = \left(\sum_{i=1}^{n} m_id_i^2\right)\dot{\theta}\mathbf{k} = I\dot{\theta}\mathbf{k},$$

where I is the moment of inertia of the body about the z-axis. This confirms the conclusion drawn from the earlier argument: when the object is symmetric about a fixed axis of rotation, the total angular momentum vector is along that axis.

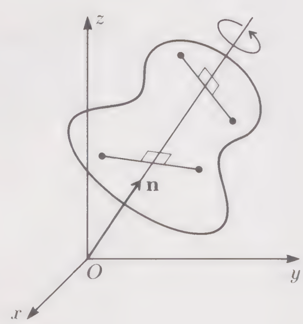


Figure 3

This conclusion applies whether the axis of rotation is the z-axis or not. Thus the body shown in Figure 3 can also be divided up into matched pairs of particles, as indicated. The particles in each pair have the same mass and are equidistant from a line fixed in the body, while the line joining the pair is perpendicular to the fixed line. Then the fixed line is known as an *axis of symmetry*, and its direction is defined by a unit vector $\mathbf{n}$. If the object spins about its axis of symmetry, which remains fixed, then we can write

This axis is known as the *spin axis* of the body.

$$\mathbf{L} = I\dot{\theta}\mathbf{n}. \tag{2}$$

The television programme used this result implicitly to describe the angular momentum of spinning tops and gyroscopes. Note that if the spin axis is not fixed then Equation (2) provides only an approximation for these cases, and the axis of symmetry of a spinning top or a gyroscope need not remain fixed. For example, Figure 4 shows a spinning top at two separate times, t_1 and t_2. The axis of symmetry of the top is defined by a unit vector, $\mathbf{n}$, which swings sideways steadily as the top wobbles. This sideways motion gives rise to a component of angular momentum perpendicular to $\mathbf{n}$, which is not included in Equation (2). However, provided that the top (or gyroscope) is spinning quickly, we can neglect the contribution to $\mathbf{L}$ caused by the sideways motion,

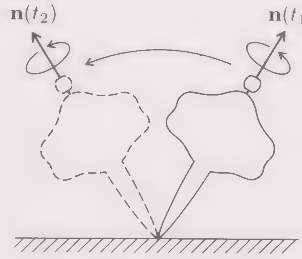


Figure 4

in comparison with the much larger component of angular momentum along $\mathbf{n}$ which is associated with the spin. The approximation is a good one provided that $|\dot{\mathbf{n}}|$ is much smaller than $\dot{\theta}$, and is certainly adequate to explain the phenomena shown in the television programme. Thus, for any rapidly spinning top or gyroscope it may be assumed that

$$\mathbf{L} \simeq I\dot{\theta}\mathbf{n}, \tag{3}$$

where $\mathbf{n}$ is a unit vector along the axis of symmetry around which the body spins.

Returning to the programme itself, demonstrations were provided of some intriguing phenomena which almost defy intuition. The first demonstration concerned the bicycle wheel and shaft described above, but spinning now with the shaft at an angle to the vertical. The wheel does not fall over, but its axis of rotation (the shaft) traces out a cone. Successive positions of the axis are shown in Figure 5. Such motion is called *precession.*

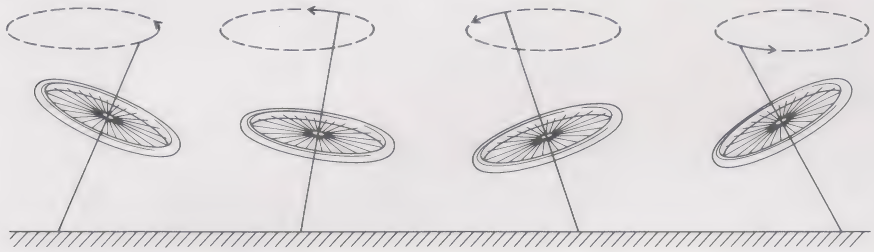


Figure 5

In the programme this behaviour was explained by use of the right hand rule and the torque law for a system of particles,

$$\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}.$$

Equation (8) of Section 1

The following exercise, which also requires the use of Equation (3), asks you to reproduce this type of argument.

Exercise 1

Suppose that, when seen from above, the axis in Figure 5 moves in an anticlockwise sense. In what sense does the wheel spin? Explain your answer.

[*Solution on page 53*]

A second demonstration is illustrated in Figure 6. This is a drawing of the gyroscope used in the programme. It consists of a rapidly rotating heavy wheel mounted in a special frame called a gimbals.

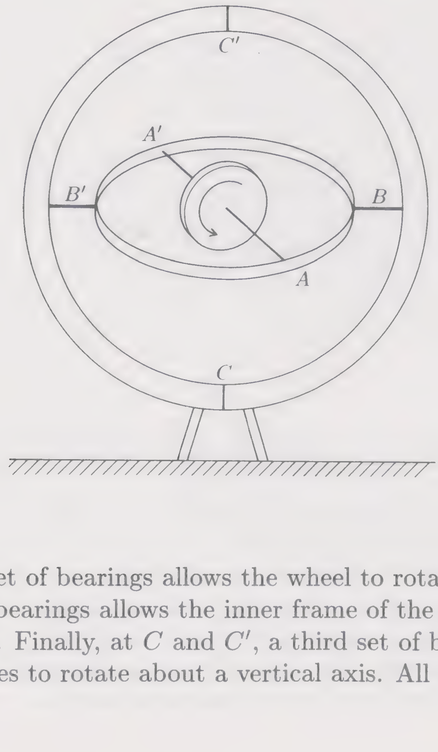


Figure 6

At points A and A' a set of bearings allows the wheel to rotate freely on its axis. At B and B', another set of bearings allows the inner frame of the gimbals to rotate freely about a horizontal axis. Finally, at C and C', a third set of bearings allows the wheel and the two inner frames to rotate about a vertical axis. All the bearings are assumed to be frictionless.

Exercise 2

In the programme, the wheel was spinning in the direction shown in Figure 6, and a weight was placed on the frame at point A. It was observed that the frame rotated about C and C' in an anticlockwise sense when viewed from above. Explain why this happened.

[*Solution on page 53*]

The final demonstration to be recalled is that of another gyroscope, whose spin axis was suspended from a frame by a pair of springs, with the frame being placed on a turntable. This apparatus is shown in Figure 7.

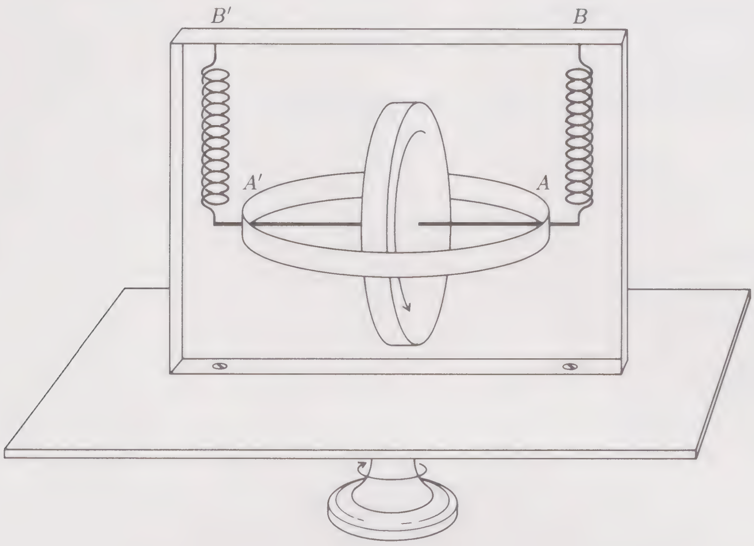


Figure 7

When the turntable rotated, one of the springs lengthened and the other shortened. This provided the torque which inevitably accompanies the rate of change of angular momentum of the gyroscope's wheel. The programme described how a similar apparatus underlies the stabilizing system of a ship.

Exercise 3

Suppose that the wheel and turntable rotate in the directions shown in Figure 7. Which of the springs lengthens? Explain your answer.

[*Solution on page 53*]

4.2 Quantitative theory of precession

In this subsection we shall use Equation (3) to make quantitative predictions about precession. The main ideas will be introduced by returning to the spinning bicycle wheel, as illustrated in Figure 8. The wheel, of mass M and radius d, spins with angular velocity $\boldsymbol{\omega} = \dot{\theta}\mathbf{n}$ around the shaft, which points in the direction of the unit vector $\mathbf{n}$ and is freely jointed at the fixed origin, O, of the coordinate system. The centre of the wheel is a distance c from O. The normal reaction at O has magnitude N.

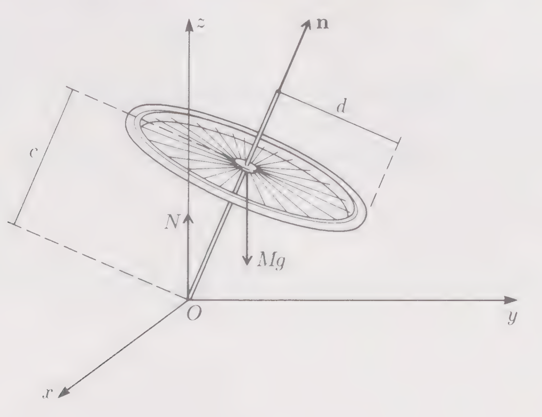


Figure 8

We shall begin by calculating the total external torque about O acting on the system. This is given by

$$\mathbf{\Gamma}^{\text{ext}} = (\mathbf{0} \times N\mathbf{k}) + (\mathbf{R} \times (-Mg\mathbf{k})) = \mathbf{R} \times (-Mg\mathbf{k}),$$

where $\mathbf{R}$ is the position of the centre of mass. Now $\mathbf{R} = c\mathbf{n}$, so that we have

$$\mathbf{\Gamma}^{\text{ext}} = -Mgc\,\mathbf{n} \times \mathbf{k} = Mgc\,\mathbf{k} \times \mathbf{n}. \tag{4}$$

This result can now be used to establish two important properties of the total angular momentum $\mathbf{L}$ of the wheel:

(i) the magnitude of $\mathbf{L}$ remains (approximately) constant;

(ii) the vertical component of $\mathbf{L}$ remains constant.

To establish the first of these properties, differentiation of $|\mathbf{L}|^2 = \mathbf{L} \cdot \mathbf{L}$ and use of the torque law $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$ gives

$$\frac{d}{dt}|\mathbf{L}|^2 = 2\mathbf{L} \cdot \dot{\mathbf{L}} = 2\mathbf{L} \cdot \mathbf{\Gamma}^{\text{ext}}.$$

Then, by Equations (3) and (4), we have

$$\frac{d}{dt}|\mathbf{L}|^2 \simeq 2I\dot{\theta}Mgc\,\mathbf{n} \cdot (\mathbf{k} \times \mathbf{n}) = 0,$$

showing that $|\mathbf{L}|$ can be taken as constant.

The second property follows in a similar way, since

$$\frac{d}{dt}(\mathbf{L} \cdot \mathbf{k}) = \dot{\mathbf{L}} \cdot \mathbf{k} = \mathbf{\Gamma}^{\text{ext}} \cdot \mathbf{k}.$$

From Equation (4), this becomes

$$\frac{d}{dt}(\mathbf{L} \cdot \mathbf{k}) = Mgc\,(\mathbf{k} \times \mathbf{n}) \cdot \mathbf{k} = 0,$$

proving that $\mathbf{L} \cdot \mathbf{k}$ is constant.

Properties (i) and (ii) of $\mathbf{L}$ are important because they greatly restrict the possible motions of the wheel. In fact, as shown in Figure 9, it must move in such a way that the tip of the arrow representing $\mathbf{L}$ traces out a circle centred on, and in a plane perpendicular to, the z-axis. This may be seen as follows. With $\mathbf{L} = L_x\mathbf{i} + L_y\mathbf{j} + L_z\mathbf{k}$, Property (i) gives

$$|\mathbf{L}|^2 = L_x^2 + L_y^2 + L_z^2 = \text{constant},$$

while Property (ii) provides

$$\mathbf{L} \cdot \mathbf{k} = L_z = |\mathbf{L}|\cos\beta = \text{constant},$$

where β is the (constant) angle between $\mathbf{L}$ and $\mathbf{k}$. Hence we have

$$L_x^2 + L_y^2 = |\mathbf{L}|^2 \sin^2\beta.$$

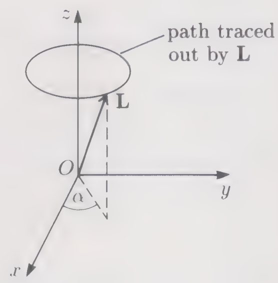


Figure 9

This shows that the tip of the vector representing $\mathbf{L}$ traces out a circle of radius $|\mathbf{L}|\sin\beta$, which is centred on the z-axis and lies in the plane $z = |\mathbf{L}|\cos\beta$.

By following a similar argument to that which led to Equation (1) of Section 2, we conclude that

$$\dot{\mathbf{L}} = \dot{\alpha}\,\mathbf{k} \times \mathbf{L},$$

Equation (1) of Section 2 stated that a particle with position vector $\mathbf{r}_i$, which moves with angular velocity $\dot{\theta}\mathbf{k}$ around a circle centred on and perpendicular to the z-axis, has velocity given by

$$\dot{\mathbf{r}}_i = \dot{\theta}\mathbf{k} \times \mathbf{r}_i.$$

Here the tip of the vector $\mathbf{L}$ undergoes the same type of motion as such a particle, with angular velocity $\dot{\alpha}\mathbf{k}$.

where α is the angle between the x-axis and the projection of $\mathbf{L}$ onto the (x, y)-plane (see Figure 9), and $\dot{\alpha}$ is called the *rate of precession* of the wheel. Using the approximation for $\mathbf{L}$ provided by Equation (3), we can write the rate of change of the total angular momentum as

$$\dot{\mathbf{L}} \simeq I\dot{\theta}\dot{\alpha}\,\mathbf{k} \times \mathbf{n}.$$

According to the torque law and Equation (4), this is equal to the total external torque

$$\boldsymbol{\Gamma}^{\text{ext}} = Mgc\,\mathbf{k} \times \mathbf{n},$$

so we deduce that

$$I\dot{\theta}\dot{\alpha} \simeq Mgc,$$

giving

$$\dot{\alpha} \simeq \frac{Mgc}{I\dot{\theta}}. \tag{5}$$

Finally, since most of the mass of a bicycle wheel is concentrated at its rim, we can assume that $I \simeq Md^2$, leading to the conclusion

$$\dot{\alpha} \simeq \frac{gc}{\dot{\theta}d^2}.$$

Exercise 4

Can Equation (5) be used when $\dot{\theta}$ is very small?

Exercise 5

The gyroscope in Figure 6 (page 42) has a homogeneous cylindrical wheel of mass M and radius R, which spins at high angular speed $|\dot{\theta}|$. When viewed from the point A, the wheel rotates in an anticlockwise sense. What is the rate of precession of the gyroscope when a weight of mass m is placed at A, which is a distance c from the static centre of the wheel?

Exercise 6

The gyroscope in Figure 7 (page 43) has a homogeneous cylindrical wheel of mass M and radius R, which spins at high angular speed $|\dot{\theta}|$. Each of the points A, A' is a distance c from the static centre of the wheel, and each of the (perfect) springs has stiffness k and natural length l_0. When the turntable is rotated at a constant rate, the lengths of the springs are l_1 and l_2 (where $l_1 > l_2$). Show that the angular speed of the turntable is given by

$$|\dot{\alpha}| \simeq \frac{2ck(l_1 - l_2)}{MR^2|\dot{\theta}|}.$$

[*Solutions on page 53*]

Summary of Section 4

1. If a body spins around an axis of symmetry (the *spin axis*), and this axis is fixed, then the total angular momentum of the body is given by

   $$\mathbf{L} = I\dot{\theta}\,\mathbf{n},$$

   where I is the moment of inertia with respect to the spin axis and $\dot{\theta}\mathbf{n}$ is the angular velocity of the body about this axis.

2. If a body spins around an axis of symmetry which is not fixed, and $I, \dot{\theta}\mathbf{n}$ are defined with respect to the spin axis as above, then the approximation

   $$\mathbf{L} \simeq I\dot{\theta}\,\mathbf{n}$$

   is valid provided that $|\dot{\mathbf{n}}|$ is small compared with $\dot{\theta}$.

3. The angular momentum of a spinning top has a constant magnitude and a constant vertical component. It satisfies the equation

   $$\dot{\mathbf{L}} = \dot{\alpha}\,\mathbf{k} \times \mathbf{L},$$

   where $\dot{\alpha}$ is the *rate of precession* of the spin axis about the vertical axis, and $\mathbf{k}$ is a unit vector in the vertically upward direction. If the top has mass M, moment of inertia I and angular velocity $\dot{\theta}\mathbf{n}$ about the spin axis, then the rate of precession is given by

   $$\dot{\alpha} \simeq \frac{Mgc}{I\dot{\theta}},$$

   where c is the distance of the centre of mass from the point of contact with the ground.

5 End of unit exercises

Section 2

Exercise 1

A man of mass m stands on a uniform horizontal disc of mass M and radius R which is initially at rest. The disc can be rotated without friction about a vertical axis through its centre. The man starts to move. What is the angular velocity of the disc when the man walks anticlockwise around a circle of radius r concentric with the disc, at speed v relative to the disc?

Exercise 2

The ends of a uniform rod of length $2a$ can slide on the inside of a smooth circular track of inner radius b, which lies in a vertical plane. Use conservation of total mechanical energy to find the angular frequency of small oscillations of the rod about its equilibrium position.

Exercise 3

A uniform rod of mass m and length $2a$ is held nearly vertical, with its lower end resting on a rough plane. It is then released from rest and falls forwards, with the inclination to the vertical at any instant being denoted by θ. At the lower end, the normal reaction has magnitude N and the frictional force has magnitude S.

(i) Use the conservation of total mechanical energy to show that

$$\dot{\theta}^2 = \frac{3g}{2a}(1-\cos\theta) \quad \text{and} \quad \ddot{\theta} = \frac{3g}{4a}\sin\theta$$

for the initial motion when the lower end of the rod remains stationary on the plane.

(ii) Use the equation $m\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}}$, where $\mathbf{R}$ is the position of the centre of mass and $\mathbf{F}^{\text{ext}}$ is the total external force on the rod, to show that

$$N = \tfrac{1}{4}mg(1-3\cos\theta)^2 \quad \text{and} \quad S = \tfrac{3}{4}mg\sin\theta\,|3\cos\theta - 2|$$

for the initial motion.

[Hint: Use the polar coordinate unit vectors $\mathbf{e}_r, \mathbf{e}_\theta$ which were introduced in *Unit 28* Subsection 4.2, and the equation

$$\ddot{\mathbf{r}} = -r\dot{\theta}^2\mathbf{e}_r + r\ddot{\theta}\mathbf{e}_\theta$$

for circular motion which was derived in that subsection. For the falling rod, $\mathbf{e}_r$ is directed along the rod (with the origin O at its lower end) and $\mathbf{e}_\theta$ points in the direction of rotation of the rod.]

(iii) Let μ be the coefficient of static friction. Show that there is a number μ_0 such that

(a) if $\mu < \mu_0$, then the rod slides backwards before θ reaches the value $\arccos\frac{2}{3}$;

(b) if $\mu \geq \mu_0$, then the above does not occur, but instead the rod slides forwards before θ reaches the value $\arccos\frac{1}{3}$.

[*Solutions on page 54*]

Section 3

Exercise 4

A uniform ball-bearing of radius a rolls back and forth without slipping on a circular track of radius b, which lies in a vertical plane. Use conservation of total mechanical energy to show that the frequency ω of small oscillations about the equilibrium position is given by

$$\omega = \sqrt{\frac{5g}{7(b-a)}}.$$

Exercise 5

A uniform ladder of mass M and length l stands on level ground and leans against a vertical wall. There is no friction at either the ground or the wall, so the ladder experiences only the forces shown in Figure 1.

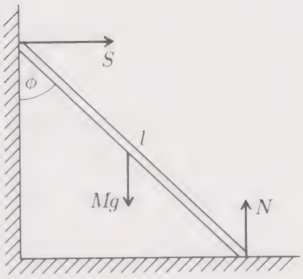


Figure 1

(i) Use the equations $M\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}}$ and $\dot{\mathbf{L}}_G = \mathbf{\Gamma}_G$ to show that the angle ϕ between the ladder and the wall satisfies the equation of motion

$$\ddot{\phi} = \frac{3g}{2l}\sin\phi.$$

(ii) Suppose that the ladder is released from rest with $\phi = \phi_0$. By multiplying the result of part (i) by $\dot{\phi}$ and integrating both sides with respect to time, show that the motion of the ladder satisfies the equation

$$\dot{\phi}^2 = \frac{3g}{l}(\cos\phi_0 - \cos\phi).$$

(iii) Use the results of parts (i) and (ii) to show that if the ladder is released from rest with $\phi_0 > 0$, then it parts company with the wall when it is at two thirds of its original height.

[*Solutions on page 55*]

Section 4

Exercise 6

Figure 2 shows a wheel of mass M and radius a which moves at constant speed v around a horizontal circle of radius R, where R is much larger than a. The wheel leans towards the centre of the circle, with a constant angle α between its plane and the upward unit vector $\mathbf{k}$.

(i) a perspective view

(ii) view from behind wheel

Figure 2

(i) Show that the total angular momentum of the wheel relative to its centre of mass is given by

$$\mathbf{L}_G \simeq Mav\,(-\cos\alpha\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) - \sin\alpha\,\mathbf{k}),$$

where θ is the angle between the x-axis and the line joining the centre of the circle to the point where the wheel touches the ground.

(ii) Show that the total external torque acting on the wheel relative to the centre of mass is

$$\mathbf{\Gamma}_G = Ma\cos\alpha\left(\frac{v^2}{R} - g\tan\alpha\right)(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}).$$

(iii) Use the results of parts (i) and (ii) to show that

$$\tan\alpha \simeq \frac{2v^2}{gR}.$$

(Compare this result with that of Exercise 9 in Section 3.)

[*Solution on page 56*]

Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

1. **(i)** Let $\mathbf{r}$ and $\mathbf{r}'$ be respectively the position vectors of a particle referred to the frames $Oxyz$ and $O'x'y'z'$, and let $\mathbf{s} = \overrightarrow{OO'}$ be the position vector of the origin O' of the second frame with respect to O.

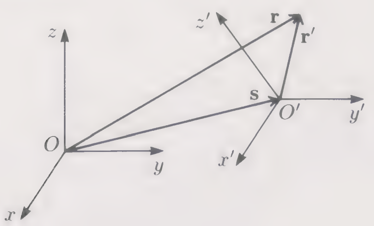


Figure 1

Note first that being at rest is a special case of motion with uniform velocity. If a particle moves with uniform velocity with respect to frame $Oxyz$, then $\dot{x}, \dot{y}, \dot{z}$ are each constant on the particle's trajectory. Similarly, uniform velocity as measured with respect to frame $O'x'y'z'$ corresponds to $\dot{x}', \dot{y}'$ and $\dot{z}'$ being constant. Since *every* point fixed in frame $O'x'y'z'$ travels with constant velocity $\mathbf{u}$ relative to O, the Cartesian unit vectors $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ of the second frame do not change with time. Hence the constancy of $\dot{x}', \dot{y}', \dot{z}'$ is equivalent to that of the vector

$$\dot{\mathbf{r}}' = \dot{x}'\mathbf{i}' + \dot{y}'\mathbf{j}' + \dot{z}'\mathbf{k}'.$$

Now, by the addition of vectors, we have

$$\mathbf{r} = \mathbf{r}' + \mathbf{s}.$$

Differentiation gives

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}' + \dot{\mathbf{s}} = \dot{\mathbf{r}}' + \mathbf{u}, \tag{1}$$

and since $\mathbf{u}$ is constant, the constancy of $\dot{\mathbf{r}}'$ follows from that of $\dot{\mathbf{r}}$. Thus if Newton's first law holds in $Oxyz$, then it also holds in $O'x'y'z'$.

(ii) A further differentiation of Equation (1) produces

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}',$$

since $\mathbf{u}$ is a constant vector. Thus if Newton's second law $\mathbf{F} = m\ddot{\mathbf{r}}$ holds in $Oxyz$ then, since the unit vectors $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ are constant, we have

$$\mathbf{F} = m\ddot{\mathbf{r}}' = m(\ddot{x}'\mathbf{i}' + \ddot{y}'\mathbf{j}' + \ddot{z}'\mathbf{k}')$$

in frame $O'x'y'z'$.

(iii)

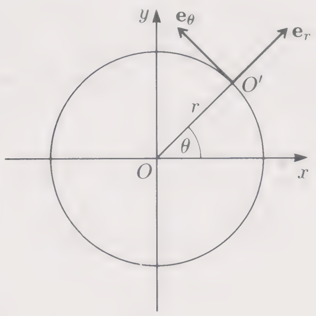


Figure 2

To show that a frame is not inertial, it suffices to establish that there is a particle on which no force acts whose motion with respect to the frame is non-uniform. Consider, for example, a particle on which no force acts, with initial position vector $\mathbf{i}$ and initial velocity $\mathbf{0}$. Since $\{\mathbf{i}, \mathbf{j}\}$ is an inertial frame, the particle remains at rest at $\mathbf{i}$. However, viewed from the frame $\{\mathbf{e}_r, \mathbf{e}_\theta\}$, this particle appears to describe a clockwise circle of radius r with centre at $-r\mathbf{e}_r$. This is a non-uniform motion (with respect to the moving frame) which is not caused by any force, so the rotating frame $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ is therefore *not* inertial.

2. **(i)** We have

$$\mathbf{F}_{12} = \lambda\mathbf{r}_{12} = \lambda(\mathbf{r}_2 - \mathbf{r}_1) = -\lambda(\mathbf{r}_1 - \mathbf{r}_2) = -\lambda\mathbf{r}_{21} = -\mathbf{F}_{21}.$$

Thus the weak form of Newton's third law holds. Since $\mathbf{r}_{12}$ and $\mathbf{r}_{21}$ are directed along the line which joins the particles, the strong form is also valid.

(ii) Clearly the strong form of the third law cannot hold here, since each of the cross products is perpendicular to the direction of $\mathbf{r}_{12}$. Now since $\mathbf{r}_{12} = -\mathbf{r}_{21}$, we also have $\dot{\mathbf{r}}_{12} = -\dot{\mathbf{r}}_{21}$. It follows that $\mathbf{F}_{12} = \mathbf{F}_{21}$, so even the weak form of Newton's third law fails to apply.

3. By definition, the angular momentum vector is

$$\begin{aligned} \boldsymbol{l} &= \mathbf{r} \times m\dot{\mathbf{r}} \\ &= (\mathbf{i} + 2\mathbf{j}) \times 3(2\mathbf{i} - \mathbf{j}) \\ &= -3\mathbf{i} \times \mathbf{j} + 12\mathbf{j} \times \mathbf{i} \\ &= -15\mathbf{k}. \end{aligned}$$

4. The angular momentum $\boldsymbol{l}$ is defined as $\mathbf{r} \times \mathbf{p}$, where $\mathbf{p} = m\dot{\mathbf{r}}$. From *Unit 28* Subsection 4.3 we have $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$, where $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$. Thus with $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, and using the rules for vector cross products from *Unit 14* Section 3, we obtain

$$\begin{aligned} \boldsymbol{l} &= \mathbf{r} \times m(\boldsymbol{\omega} \times \mathbf{r}) \\ &= (x\mathbf{i} + y\mathbf{j}) \times m\left(\dot{\theta}\mathbf{k} \times (x\mathbf{i} + y\mathbf{j})\right) \\ &= m\dot{\theta}(x\mathbf{i} + y\mathbf{j}) \times (x\mathbf{j} - y\mathbf{i}) \\ &= m\dot{\theta}(x^2 + y^2)\mathbf{k} = md^2\dot{\theta}\mathbf{k}. \end{aligned}$$

This outcome can also be obtained by putting $z = 0$, $\omega = \dot{\theta}$ and $x^2 + y^2 = d^2$ into the result of Example 2.

Another alternative is to use the polar coordinate frame $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k}\}$ introduced in *Unit 28* Section 4. In terms of this frame, we have

$$\mathbf{r} = d\mathbf{e}_r, \quad \mathbf{p} = m\dot{\mathbf{r}} = md\dot{\mathbf{e}}_r = md\dot{\theta}\mathbf{e}_\theta,$$

and hence

$$\boldsymbol{l} = \mathbf{r} \times \mathbf{p} = md^2\dot{\theta}\mathbf{e}_r \times \mathbf{e}_\theta = md^2\dot{\theta}\mathbf{k}.$$

5. By the torque law for a system of n particles, the total angular momentum of the system will have a constant z-component L_z provided that the total torque acting on the system has a zero z-component, that is, provided that

$$\Gamma_z^{\text{ext}} = \mathbf{\Gamma}^{\text{ext}} \cdot \mathbf{k} = 0.$$

For the ith particle of the system, the gravitational torque about an arbitrary origin is

$$\boldsymbol{\gamma}_i^{\text{ext}} = \mathbf{r}_i \times (-m_i g\mathbf{k}),$$

so that $\boldsymbol{\gamma}_i^{\text{ext}} \cdot \mathbf{k} = 0$. Thus

$$\mathbf{\Gamma}^{\text{ext}} \cdot \mathbf{k} = \left(\sum_{i=1}^{n} \boldsymbol{\gamma}_i^{\text{ext}}\right) \cdot \mathbf{k} = \sum_{i=1}^{n} \left(\boldsymbol{\gamma}_i^{\text{ext}} \cdot \mathbf{k}\right) = 0,$$

and L_z is constant.

6. We may use the results of Example 4, since the form of the inter-particle force (gravitational here, a spring force in Example 4) plays no role in the torque law as long as the strong form of Newton's third law is satisfied (as is the case for the gravitational force). Thus the (constant) total angular momentum of the star system has magnitude

$$L = \frac{m_1 m_2 d^2 \omega}{m_1 + m_2},$$

where ω is the constant angular speed of each star about the centre of mass. The period of revolution is therefore

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi m_1 m_2 d^2}{(m_1 + m_2)L}.$$

7. As every particle is at rest, the total angular momentum $\mathbf{L}$ of the system is certainly zero. Using the torque law $\dot{\mathbf{L}} = \mathbf{\Gamma}^{\text{ext}}$, the total external torque acting on the system is

$$\mathbf{\Gamma}^{\text{ext}} = \mathbf{0},$$

which is the result applied in *Unit 28*.

Solutions to the exercises in Section 2

1. **(i)** From the definition of moment of inertia, we have

$$I = \sum_{i=1}^{2} m_i d_i^2,$$

where $m_1 = m_2 = m$ and $d_1 = d_2 = d$. Hence $I = 2md^2$.

(ii) The kinetic energy of rotation about the specified axis is

$$\tfrac{1}{2}I\dot{\theta}^2 = md^2\dot{\theta}^2.$$

[In this simple case one can also see from first principles that the kinetic energy is

$$\tfrac{1}{2}md^2\dot{\theta}^2 + \tfrac{1}{2}md^2\dot{\theta}^2 = md^2\dot{\theta}^2.]$$

2. Since friction is negligible, the law of conservation of mechanical energy may be used. The datum for gravitational potential energy is chosen to be at the bottom of the shaft. Then when the lift is at the top of the shaft, the total mechanical energy is

$$E = Mgh,$$

while at the bottom of the shaft it is

$$E = \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}Mv^2$$

(the sum of the rotational kinetic energy of the flywheel and the translational kinetic energy of the lift). The speed v of the lift must equal the speed of the rim of the flywheel, which may also be expressed as $R\dot{\theta}$. Hence $\dot{\theta} = v/R$, and on equating the two expressions above for E, we find that

$$Mgh = \tfrac{1}{2}Iv^2/R^2 + \tfrac{1}{2}Mv^2.$$

Solving this equation for v gives

$$v = \sqrt{\frac{2Mgh}{M + I/R^2}}.$$

3. Using the method of Example 1, we have

$$I = \int_0^h \rho_0 x^2\,dx = \tfrac{1}{3}\rho_0 h^3.$$

As in the example, the linear density is $\rho_0 = M/h$, so the moment of inertia of the rod is

$$I = \tfrac{1}{3}Mh^2.$$

4. The linear mass density along the hoop is $\rho_0 = M/(2\pi R)$, and a small section of hoop subtending an angle $\delta\theta$ at the centre has length $R\,\delta\theta$ and mass $\rho_0 R\,\delta\theta$. The contribution of this section to the moment of inertia is $(\rho_0 R\,\delta\theta)R^2$, so the hoop has moment of inertia

$$I = \int_0^{2\pi} \rho_0 R^3\,d\theta = 2\pi\rho_0 R^3 = MR^2.$$

5. We may use the method of Example 2, where now the integral for I has r-limits a and R. The surface density ρ_0 here is given by

$$\pi(R^2 - a^2)\rho_0 = M,$$

and the moment of inertia is

$$\begin{aligned} I &= \int_{\theta=0}^{2\pi}\int_{r=a}^{R} \rho_0 r^3\,dr\,d\theta \\ &= \tfrac{1}{2}\pi\rho_0(R^4 - a^4) \\ &= \tfrac{1}{2}M\frac{R^4 - a^4}{R^2 - a^2} = \tfrac{1}{2}M(R^2 + a^2). \end{aligned}$$

[Note that when $a = 0$ we obtain the result for a disc (Example 2), and when $a = R$ we obtain that for the hoop of Exercise 4.]

6. **(i)** We take the x-, y- and z-axes to be parallel respectively to the sides of lengths a, b and c, with the origin at the centre of the brick. The volume of the brick is abc, so that its uniform density is $M/(abc)$. The moment of inertia about AA' (the z-axis) is therefore

$$\begin{aligned} I &= \frac{M}{abc}\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} (x^2 + y^2)\,dx\,dy\,dz \\ &= \frac{M}{abc}\left(\tfrac{2}{3}(\tfrac{1}{2}a)^3 bc + \tfrac{2}{3}(\tfrac{1}{2}b)^3 ac\right) \\ &= \tfrac{1}{12}M(a^2 + b^2). \end{aligned}$$

(ii) Here we choose the x-, y- and z-axes to be parallel respectively to the sides of length a, b and c, as in part (i), but now the origin is chosen in the centre of the face across which BB' passes (so that BB' is the z-axis). The calculation then differs from that in part (i) only in respect of the limits of integration for x. The moment of inertia about BB' is

$$\begin{aligned} I &= \frac{M}{abc}\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a}^{0} (x^2 + y^2)\,dx\,dy\,dz \\ &= \frac{M}{abc}\left(\tfrac{1}{3}a^3 bc + \tfrac{2}{3}(\tfrac{1}{2}b)^3 ac\right) \\ &= \tfrac{1}{12}M(4a^2 + b^2). \end{aligned}$$

7. For the moment of inertia about the parallel axis passing through the centre of mass, we can use the result for a solid cylinder in the fifth row of Table 1, taking the limit as the length h tends to zero to obtain

$$I_G = \tfrac{1}{4}MR^2.$$

The given axis is a distance $d = R$ from the centre of mass, so by the parallel axes theorem, we have

$$I = I_G + Md^2 = \tfrac{1}{4}MR^2 + MR^2 = \tfrac{5}{4}MR^2.$$

8. Let the moments of inertia be denoted by

I_1 for the upright cylinder,

I_2 for a horizontal arm,

I_3 for a vertical arm,

each taken about the skater's axis of rotation, which is the axis of the upright cylinder. Let ω_0 and ω_1 be the vertical components of the skater's angular velocity before and after raising her arms. Then the conservation of angular momentum gives

$$(I_1 + 2I_2)\omega_0 = (I_1 + 2I_3)\omega_1$$

or $$\omega_1 = \left(\frac{I_1 + 2I_2}{I_1 + 2I_3}\right)\omega_0 = \left(\frac{1 + 2I_2/I_1}{1 + 2I_3/I_1}\right)\omega_0.$$

To calculate the relevant moments of inertia, let M, R and H be the mass, radius and height of the cylinder representing the body, and let m, r and h be the corresponding quantities for a cylinder representing an arm. Then, using results from Table 1 of Section 2 on page 21, and applying the parallel axes theorem, we find that

$$\begin{aligned} I_1 &= \tfrac{1}{2}MR^2, \\ I_2 &= (\tfrac{1}{4}mr^2 + \tfrac{1}{12}mh^2) + m(\tfrac{1}{2}h + R)^2, \\ I_3 &= \tfrac{1}{2}mr^2 + m(r + R)^2. \end{aligned}$$

Thus,

$$\frac{I_2}{I_1} = \frac{m}{M}\left[\tfrac{1}{2}\left(\frac{r}{R}\right)^2 + \tfrac{1}{6}\left(\frac{h}{R}\right)^2 + 2\left(1 + \frac{h}{2R}\right)^2\right]$$

and

$$\frac{I_3}{I_1} = \frac{m}{M}\left[\left(\frac{r}{R}\right)^2 + 2\left(1 + \frac{r}{R}\right)^2\right].$$

The assumption that the cylinders are of the same uniform density gives

$$\frac{m}{M} = \frac{\pi r^2 h}{\pi R^2 H} = \frac{r^2 h}{R^2 H}.$$

Substituting the data given in the problem, the answer to three significant figures is

$$\omega_1 = 2.53\omega_0.$$

Thus the angular speed of the skater increases by a factor of about 2.5, in spite of the fact that her arms provide less than a tenth of the total mass. This large effect arises because the moment of inertia of a particle increases as the *square* of the particle's distance from the axis of rotation.

9. **(i)** Immediately before impact, the total linear momentum of the system comprising bat and ball is

$$\mathbf{P}_{\text{before}} = mu\mathbf{j}.$$

Immediately after impact, the total linear momentum is

$$\mathbf{P}_{\text{after}} = mv(-\mathbf{j}) + M\dot{\mathbf{R}},$$

where $\mathbf{R}$ is the position vector of the bat's centre of mass. Since the bat pivots about O, immediately after impact we have $\dot{\mathbf{R}} = \frac{1}{2}l\dot{\theta}\mathbf{j}$, so that

$$\mathbf{P}_{\text{after}} = -mv\mathbf{j} + \tfrac{1}{2}Ml\dot{\theta}\mathbf{j}.$$

Thus the law of conservation of total linear momentum for horizontal motion gives

$$mu = -mv + \tfrac{1}{2}Ml\dot{\theta}.$$

(ii) The initial angular momentum of the cricket ball is $mub\mathbf{k}$. Thus immediately before impact, the z-component of the total angular momentum of the system comprising bat and ball is

$$(L_z)_{\text{before}} = mub.$$

Immediately after impact, the z-component of the total angular momentum is

$$(L_z)_{\text{after}} = -mvb + I\dot{\theta},$$

where I is the moment of inertia of the bat about the z-axis. The bat is modelled by a thin blade of length l and mass M, so that on using Table 1 of Section 2 on page 22, and the parallel axes theorem, we obtain

$$I = \tfrac{1}{12}Ml^2 + M\left(\tfrac{1}{2}l\right)^2 = \tfrac{1}{3}Ml^2.$$

Thus, the z-component of the law of conservation of angular momentum gives

$$mub = -mvb + \tfrac{1}{3}Ml^2\dot{\theta}.$$

(iii) Combining the results of parts (i) and (ii), we see that

$$mu + mv = \tfrac{1}{2}Ml\dot{\theta} = \frac{Ml^2\dot{\theta}}{3b}$$

or $\quad b = \frac{2}{3}l.$

10.

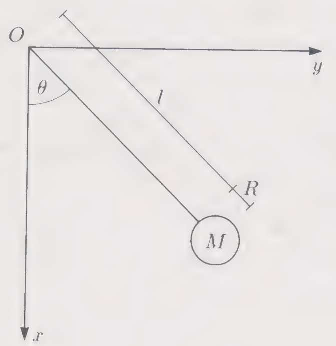


Figure 1

The diagram above shows the pendulum and the chosen coordinate system, where the z-axis is the axis of rotation. The moment of inertia of the pendulum is found by neglecting the mass of the rod and using the parallel axes theorem. This gives

$$I = \tfrac{2}{5}MR^2 + M(l+R)^2.$$

The pendulum experiences two external forces, due respectively to gravity and to the reaction at the pivot. The reaction produces no torque, because it acts at the origin.

Using Equation (11) of Section 2, and noting that here $\mathbf{i}$ is a unit vector directed vertically downwards, the gravitational force produces the torque

$$\begin{aligned}\boldsymbol{\Gamma} &= (l+R)(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \times Mg\mathbf{i}\\ &= -Mg(l+R)\sin\theta\,\mathbf{k} = \Gamma\mathbf{k}.\end{aligned}$$

From the torque law $I\ddot{\theta} = \Gamma$ (Equation (10) of Section 2), we then have

$$\ddot{\theta} + \frac{Mg(l+R)}{\frac{2}{5}MR^2 + M(l+R)^2}\sin\theta = 0.$$

For small oscillations the approximation $\sin\theta \simeq \theta$ may be used, leading to

$$\ddot{\theta} + \frac{g(l+R)}{\frac{2}{5}R^2 + (l+R)^2}\theta \simeq 0.$$

This equation describes simple harmonic oscillations of period

$$\tau \simeq 2\pi\sqrt{\frac{\frac{2}{5}R^2 + (l+R)^2}{g(l+R)}}.$$

[Note that on taking the limit of this expression as $R \to 0$ (which corresponds to replacing the sphere by a particle), we obtain $\tau \simeq 2\pi\sqrt{l/g}$, as found for the model pendulum in *Unit 7* Subsection 4.3.]

11. The kinetic energy of the pendulum is $\frac{1}{2}I\dot{\theta}^2$ where, as in the solution to Exercise 10,

$$I = \tfrac{2}{5}MR^2 + M(l+R)^2.$$

The gravitational potential energy is

$$U = -Mg(l+R)\cos\theta,$$

where $\theta = \frac{1}{2}\pi$ is taken as the datum. Thus the total mechanical energy is

$$E = \tfrac{1}{2}I\dot{\theta}^2 - Mg(l+R)\cos\theta.$$

The reaction force at the pivot does not contribute to E or affect the conservation of mechanical energy, because the point at which it acts is fixed. Hence E is constant, and differentiation of the expression for E followed by division by $\dot{\theta}$ gives

$$I\ddot{\theta} + Mg(l+R)\sin\theta = 0.$$

The solution then proceeds as for Exercise 10 above.

12. Let the vertical line through the hinges of the door be the z-axis. Then, by the torque law, we have

$$I\ddot{\theta} = \Gamma = bP,$$

so that

$$\ddot{\theta} = \frac{bP}{I} = A,$$

where A is a constant. Successive integrations give

$$\begin{aligned}\dot{\theta} &= At + B,\\ \theta &= \tfrac{1}{2}At^2 + Bt + C,\end{aligned}$$

where B and C are constants. Using the initial conditions $\theta = 0$ and $\dot{\theta} = 0$ at $t = 0$, we find $B = C = 0$. Hence

$$\dot{\theta} = At = \sqrt{2A\theta} = \sqrt{\frac{2bP\theta}{I}}.$$

When the door has swung through $\frac{1}{2}\pi$ rad, the angular speed is

$$\dot{\theta} = \sqrt{\frac{\pi bP}{I}}$$

and the speed of the handle is

$$v = b\dot{\theta} = \sqrt{\frac{\pi b^3 P}{I}}.$$

However, the moment of inertia of the door about a vertical axis through its hinges is (by the same argument as in Example 1 of the audio-tape)

$$I = \tfrac{1}{12}Mb^2 + M(\tfrac{1}{2}b)^2 = \tfrac{1}{3}Mb^2,$$

so we conclude that

$$v = \sqrt{\frac{3\pi bP}{M}}.$$

13.

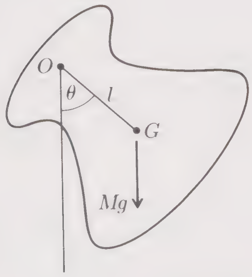


Figure 2

The lamina is shown in the diagram above. Proceeding as for the pendulum in Exercise 10, we have (from the torque law)

$$I\ddot{\theta} = \Gamma = -Mgl\sin\theta,$$

where I is the moment of inertia about an axis through O and perpendicular to the plane of the lamina. From the parallel axes theorem, I is given by

$$I = I_G + Ml^2.$$

Hence, on making the approximation $\sin\theta \simeq \theta$ for small values of θ, we obtain

$$\ddot{\theta} + \frac{Mgl}{I_G + Ml^2}\theta = 0,$$

which represents simple harmonic motion with period

$$\tau = 2\pi\sqrt{\frac{I_G + Ml^2}{Mgl}},$$

as required.

Solutions to the exercises in Section 3

1. Choosing a coordinate system as shown in the diagram below, the external force on the ith particle (of mass m_i) is

$$\mathbf{F}_i^{\text{ext}} = -m_i g\mathbf{k},$$

and the total external force is

$$\mathbf{F}^{\text{ext}} = \sum_{i=1}^{n}(-m_i g\mathbf{k}) = -\left(\sum_{i=1}^{n} m_i\right) g\mathbf{k} = -Mg\mathbf{k},$$

where M is the total mass of the n-particle system.

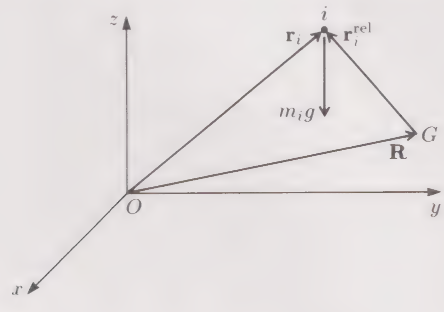


Figure 1

Then the total external torque is

$$\begin{aligned}\mathbf{\Gamma}^{\text{ext}} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} &= \sum_{i=1}^{n} \mathbf{r}_i \times (-m_i g\mathbf{k}) \\ &= -\left(\sum_{i=1}^{n} m_i \mathbf{r}_i\right) \times g\mathbf{k} \\ &= -M\mathbf{R} \times g\mathbf{k},\end{aligned}$$

by definition of the centre of mass position $\mathbf{R}$. Thus we have

$$\mathbf{\Gamma}^{\text{ext}} = \mathbf{R} \times (-Mg\mathbf{k}) = \mathbf{R} \times \mathbf{F}^{\text{ext}},$$

as required. [Alternatively, one can establish the result from the decomposition theorem by showing that $\mathbf{\Gamma}_G = \mathbf{0}$. This follows because

$$\begin{aligned}\mathbf{\Gamma}_G = \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times \mathbf{F}_i^{\text{ext}} &= \sum_{i=1}^{n} (\mathbf{r}_i - \mathbf{R}) \times (-m_i g\mathbf{k}) \\ &= \sum (m_i\mathbf{r}_i - m_i\mathbf{R}) \times (-g\mathbf{k}) \\ &= (M\mathbf{R} - M\mathbf{R}) \times (-g\mathbf{k}) = \mathbf{0}.]\end{aligned}$$

2. The total external torque $\mathbf{\Gamma}^{\text{ext}}$ is given by

$$\mathbf{\Gamma}^{\text{ext}} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}.$$

Here we have $\mathbf{F}_i^{\text{ext}} = \lambda\mathbf{r}_i$, so that

$$\mathbf{\Gamma}^{\text{ext}} = \sum_{i=1}^{n} \mathbf{r}_i \times \lambda\mathbf{r}_i = \mathbf{0}.$$

By the decomposition theorem (Equation (5) of Section 3), it follows that

$$\mathbf{\Gamma}_G = -\mathbf{R} \times \mathbf{F}^{\text{ext}}.$$

3. The disc and attached particles are shown in the figure below.

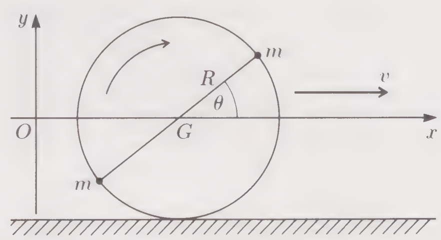


Figure 2

Since the total mass of the system is $M + 2m$, the decomposition theorem gives the total kinetic energy T as

$$T = T_G + \tfrac{1}{2}(M + 2m)v^2.$$

The kinetic energy relative to the centre of mass G is

$$T_G = \tfrac{1}{2}I\dot{\theta}^2 + 2 \times \tfrac{1}{2}mR^2\dot{\theta}^2,$$

where I is the moment of inertia of the disc about the axis of rotation through G, and $\dot{\theta}$ is the angular velocity about this axis (which is negative for the case shown above). From Example 2 of Section 2, we have $I = \frac{1}{2}MR^2$, and the rolling condition gives $-R\dot{\theta} = v$. It follows that

$$T_G = \tfrac{1}{4}Mv^2 + mv^2,$$

from which we obtain

$$\begin{aligned}T &= (\tfrac{1}{4}M + m)v^2 + \tfrac{1}{2}(M + 2m)v^2 \\ &= (\tfrac{3}{4}M + 2m)v^2,\end{aligned}$$

as required.

4. As in Example 1, we start by writing down an expression for the total mechanical energy $E = T + U$ of the system, taking the direction of the x-axis to be down the slope as before.

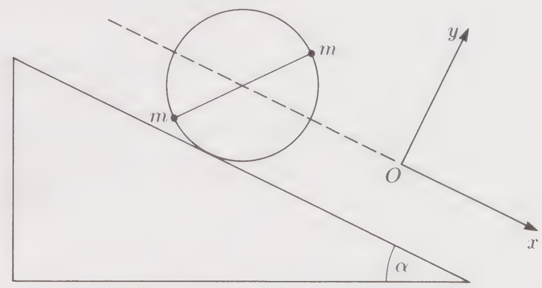


Figure 3

The kinetic energy T was obtained in Exercise 3. With v replaced by $\dot{X}$, it is

$$T = (\tfrac{3}{4}M + 2m)\dot{X}^2.$$

The gravitational potential energy is given, as in Equation (7) of Section 3, by the height (above the origin) of the centre of mass, $-X \sin\alpha$, times the total mass times g, that is,

$$U = -(M + 2m)Xg\sin\alpha.$$

The total mechanical energy is therefore

$$E = (\tfrac{3}{4}M + 2m)\dot{X}^2 - (M + 2m)Xg\sin\alpha.$$

Conservation of total mechanical energy then gives $\dot{E} = 0$, which is equivalent to

$$2(\tfrac{3}{4}M + 2m)\dot{X}\ddot{X} - (M + 2m)\dot{X}g\sin\alpha = 0.$$

Hence we have

$$\ddot{X} = \left(\frac{M + 2m}{\frac{3}{2}M + 4m}\right) g\sin\alpha.$$

5. Let the x-axis point directly down the slope, and let X be the x-coordinate of the centre of mass of the steamroller. Then, using Equation (6) of Section 3 for the kinetic energy of each roller, the total kinetic energy of the steamroller is

$$T = \tfrac{1}{2}M\dot{X}^2 + \tfrac{3}{4}M_1\dot{X}^2 + \tfrac{3}{4}M_2\dot{X}^2.$$

Using Equation (7) of Section (3), the potential energy of the steamroller is

$$U = -(M + M_1 + M_2)Xg\sin\alpha.$$

Thus the total mechanical energy is

$$E = \tfrac{1}{2}(M + \tfrac{3}{2}M_1 + \tfrac{3}{2}M_2)\dot{X}^2 - (M + M_1 + M_2)Xg\sin\alpha.$$

Since E is constant, differentiation with respect to time and division by $\dot{X}$ gives

$$\ddot{X} = \left(\frac{M + M_1 + M_2}{M + \frac{3}{2}M_1 + \frac{3}{2}M_2}\right) g\sin\alpha.$$

6. (i) We first proceed as in Example 1. The total kinetic energy is given by

$$T = \tfrac{1}{2}M\dot{X}^2 + T_G = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}I\dot{\theta}^2,$$

where M is the mass of the hollow cylinder and I is the moment of inertia of the cylinder about its central axis. From Table 1 of Section 2 on page 21, we have $I = MR^2$, since the cylinder is thin-walled. Also, the condition for rolling is $\dot{X} = -R\dot{\theta}$, so the total kinetic energy is

$$T = M\dot{X}^2.$$

The gravitational potential energy is given (with datum at the origin) by

$$U = -MXg\sin\alpha,$$

and the total mechanical energy E is therefore given by

$$E = T + U = M\dot{X}^2 - MXg\sin\alpha.$$

Conservation of total mechanical energy gives $\dot{E} = 0$, leading with $\alpha = \frac{1}{4}\pi$ to

$$\ddot{X} = \tfrac{1}{2}g\sin\alpha = \frac{g}{2\sqrt{2}} \simeq 3.47\,\mathrm{m\,s^{-2}}.$$

(ii) As in Example 2, and from part (i), the x-component of the total external force is equal to

$$M\ddot{X} = \tfrac{1}{2}Mg\sin\alpha.$$

Resolving forces in the x-direction gives, as in Example 2,

$$M\ddot{X} = Mg\sin\alpha - S,$$

where S is the magnitude of the frictional force, so that

$$S = Mg\sin\alpha - \tfrac{1}{2}Mg\sin\alpha = \tfrac{1}{2}Mg\sin\alpha.$$

Also as in Example 2, we have

$$N = Mg\cos\alpha,$$

from which it follows that

$$\frac{S}{N} = \tfrac{1}{2}\tan\alpha = \tfrac{1}{2}.$$

Now in the absence of slipping, this ratio must be less than or equal to the coefficient of static friction μ, that is, $\frac{1}{2} \le \mu$. The cylinder will slide if $\mu < \frac{1}{2}$.

[Alternatively, the result may be obtained immediately using a result from Example 2, namely, that the cylinder will roll without slipping only if

$$\tan\alpha \le \left(\frac{1+\beta}{\beta}\right)\mu.$$

In this case $\beta = I/(MR^2) = 1$ and $\tan\alpha = 1$, whence the result follows.]

7. As in the solution to part (iii) of Example 2, the acceleration of the centre of mass of a rigid body of circular cross-section, rolling down a plane which is inclined at an angle α to the horizontal, is given by

$$\ddot{X} = \frac{g\sin\alpha}{1+\beta},$$

where $\beta = I/(MR^2)$, I is the moment of inertia of the rigid body about the rolling axis through its centre of mass, M is the total mass and R is the radius. For a solid sphere, the value of β is $\beta_S = \frac{2}{5}$, while for a hollow sphere of negligible thickness its value is $\beta_H = \frac{2}{3}$ (from Table 1 of Section 2 on page 21). The distance travelled from rest in time t is

$$\tfrac{1}{2}t^2\frac{g\sin\alpha}{1+\beta}.$$

Thus the ratio of the distances travelled by the solid and hollow spheres in a given time interval is

$$\frac{1+\beta_H}{1+\beta_S} = \frac{1+\frac{2}{3}}{1+\frac{2}{5}} = \frac{25}{21}.$$

8. According to Equation (10) of Section 3, we have

$$I\ddot{\theta} = \Gamma_G,$$

where $\mathbf{\Gamma}_G = \Gamma_G\mathbf{k}$ is the total external torque relative to the centre of mass. However, since the only external force acting on the spanner is due to gravity, we may use the result of Exercise 1 to deduce that $\mathbf{\Gamma}_G = \mathbf{0}$. It follows that $\ddot{\theta} = 0$ and hence that $\dot{\theta}$ is constant. The angle θ between the spanner and the horizontal therefore varies at a constant rate.

9. (i)

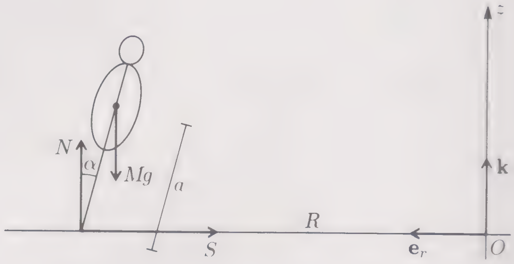


Figure 4

The diagram above shows the forces acting on the skater, which are the downward gravitational force of magnitude Mg, an upward normal reaction force of magnitude N, and a frictional force of magnitude S directed towards the centre of the circle (to produce the inward acceleration). Taking the origin to be at the centre of the circle, with the z-axis vertically upwards, and using the outward radial unit vector $\mathbf{e}_r$ in the (x, y)-plane, the total external force acting on the skater is

$$\begin{aligned}\mathbf{F}^{\text{ext}} &= Mg(-\mathbf{k}) + N\mathbf{k} + S(-\mathbf{e}_r)\\ &= -S\mathbf{e}_r + (N - Mg)\mathbf{k}.\end{aligned}$$

However, the skater moves around a circle with constant speed v, and inward acceleration of constant magnitude v^2/r, so we have

$$\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}} = \frac{Mv^2}{R}(-\mathbf{e}_r).$$

It follows that

$$S = \frac{Mv^2}{R} \quad \text{and} \quad N = Mg.$$

Now let the distance of the skates from the centre of mass be a. Then the position vector of the skates relative to the centre of mass is

$$a\cos\alpha(-\mathbf{k}) + a\sin\alpha\,\mathbf{e}_r.$$

Thus the total external torque relative to the centre of mass is

$$\begin{aligned}\mathbf{\Gamma}_G &= (-a\cos\alpha\,\mathbf{k} + a\sin\alpha\,\mathbf{e}_r) \times \left(-\frac{Mv^2}{R}\mathbf{e}_r + Mg\mathbf{k}\right)\\ &= \left(a\cos\alpha\,\frac{Mv^2}{R} - a\sin\alpha\,Mg\right)\mathbf{k}\times\mathbf{e}_r\\ &= Ma\cos\alpha\left(\frac{v^2}{R} - g\tan\alpha\right)\mathbf{e}_\theta,\end{aligned}$$

where $\mathbf{e}_\theta$ is a unit vector in the direction of motion of the skater.

(ii) If all the particles in the skater are assumed to have the same velocity, then each particle has the velocity $\dot{\mathbf{R}}$ of the centre of mass, so that the velocity $\dot{\mathbf{r}}_i^{\text{rel}}$ of any particle relative to the centre of mass is zero. Then the definition of the total angular momentum relative to the centre of mass (see the statement of the centre of mass decomposition theorems) gives

$$\begin{aligned}\mathbf{L}_G &= \sum_{i=1}^{n} \mathbf{r}_i^{\text{rel}} \times m_i\dot{\mathbf{r}}_i^{\text{rel}}\\ &= \sum_{i=1}^{n} m_i\mathbf{r}_i^{\text{rel}} \times \mathbf{0} = \mathbf{0}.\end{aligned}$$

It follows that $\dot{\mathbf{L}}_G = \mathbf{0}$ and hence, from the torque law relative to the centre of mass, that $\mathbf{\Gamma}_G = \mathbf{0}$. The result of part (i) then gives

$$\tan\alpha = \frac{v^2}{gR}.$$

Solutions to the exercises in Section 4

1. The wheel spins in an anticlockwise sense when viewed from above. To explain this, consider the diagram below.

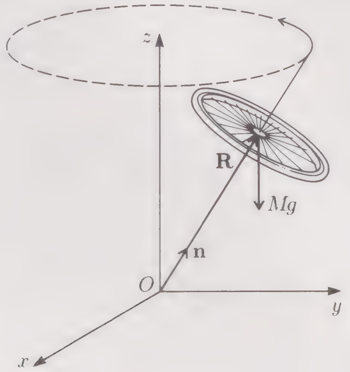


Figure 1

The total external torque relative to the point of support O is

$$\mathbf{\Gamma}^{\text{ext}} = \mathbf{R} \times Mg(-\mathbf{k})$$

which, by the right hand rule, points into the page. Hence, by the torque law, $\dot{\mathbf{L}}$ also points into the page. However, we know that the precession (movement of the axis) appears anticlockwise when viewed from above, so if $\mathbf{n}$ is a unit vector pointing along the shaft from the origin towards the wheel, then $\dot{\mathbf{n}}$ points into the page. Since the rate of spin $\dot{\theta}$ is constant, differentiation of Equation (3) of Section 4 gives

$$\dot{\mathbf{L}} \simeq I\dot{\theta}\dot{\mathbf{n}}.$$

The directions of $\dot{\mathbf{L}}$ and $\dot{\mathbf{n}}$ are the same, so we conclude that $\dot{\theta}$ is positive. According to our sign convention, this means that the wheel spins anticlockwise when viewed from above.

2. The diagram of the gyroscope and weight (of mass m) is shown below.

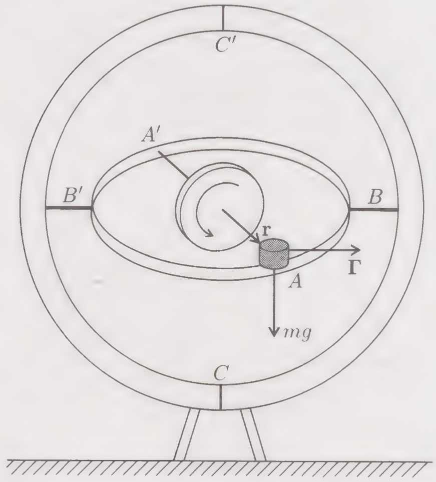


Figure 2

The wheel spins in an anticlockwise sense when viewed from the weight so, from Equation (3) of Section 4, the angular momentum vector points out of the page.

Choosing the origin to be at the static centre of the wheel and the z-axis to be vertically upwards, the right hand rule establishes that the torque

$$\mathbf{\Gamma} = \mathbf{r} \times mg(-\mathbf{k})$$

points in the direction indicated. By the torque law, $\dot{\mathbf{L}}$ must also point in this direction, so we conclude that the gyroscope precesses about a vertical axis, in an anticlockwise sense when seen from above.

3. Referring to Figure 7 of Section 4, both springs exert upward forces on the inner frame of the gyroscope (as they support the gyroscope's weight). Thus, by the right hand rule, the torque exerted by the spring AB (relative to the centre of the wheel) points out of the page, while that exerted by the spring $A'B'$ points into the page. Now, the angular momentum of the wheel is directed from A' to A, so the rotation of the turntable causes $\dot{\mathbf{L}}$ to point out of the page. By the torque law, the total external torque is also directed out of the page, so spring AB must exert a torque of greater magnitude than spring $A'B'$. Spring AB therefore exerts more force, and lengthens, while spring $A'B'$ shortens.

4. Equation (5) of Section 4 cannot be expected to be valid when $\dot{\theta}$ is very small, because Equation (3) of Section 4 is a good approximation only when $\dot{\theta}$ is much greater than $|\dot{\mathbf{n}}|$.

5. The static centre of mass of the wheel is chosen as the origin, with the z-axis pointing vertically upwards. Then the position vector of the weight is

$$\mathbf{r} = c\mathbf{n},$$

where $\mathbf{n}$ is a unit vector along the axis of rotation of the wheel and in the same direction as the angular momentum $\mathbf{L}$. Proceeding as in the solution to Exercise 2, the total external torque acting on the gyroscope is

$$\mathbf{\Gamma}^{\text{ext}} = \mathbf{r} \times mg(-\mathbf{k}) = mgc\,\mathbf{k} \times \mathbf{n}.$$

This has exactly the same form as Equation (4) of Section 4, so the argument given in the text may be repeated to obtain the equivalent of Equation (5) of Section 4, which is

$$\dot{\alpha} \simeq \frac{mgc}{I\dot{\theta}}.$$

The moment of inertia of the solid cylindrical wheel is

$$I = \tfrac{1}{2}MR^2,$$

so we conclude that the rate of precession of the gyroscope is

$$\dot{\alpha} \simeq \frac{2mgc}{MR^2\dot{\theta}}.$$

6. We again choose the static centre of the wheel as the origin, with the z-axis pointing vertically upwards. Let T_1 be the tension in spring AB and T_2 the tension in spring $A'B'$, where $T_1 > T_2$ from the solution to Exercise 3. The unit vector $\mathbf{n}$ points along the spin axis in the direction of the angular momentum. The diagram below corresponds to an instant when the spin axis lies in the (y, z)-plane.

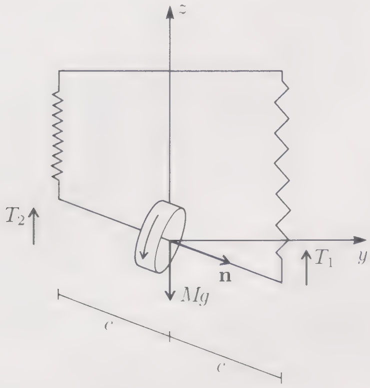


Figure 3

At this time the total external torque relative to the centre of the wheel is

$$\begin{aligned}\mathbf{\Gamma}^{\text{ext}} &= (c\mathbf{n} \times T_1\mathbf{k}) + (c(-\mathbf{n}) \times T_2\mathbf{k}) \\ &= c(T_1 - T_2)\,\mathbf{n} \times \mathbf{k}.\end{aligned}$$

This is similar to Equation (4) of Section 4, so that on repeating the argument given in the text, we obtain

$$\dot{\mathbf{L}} = \dot{\alpha}\,\mathbf{k} \times \mathbf{L},$$

where $\dot{\alpha}$ is the rate of precession. It follows from Equation (3) of Section 4 that

$$\dot{\mathbf{L}} \simeq I\dot{\theta}\dot{\alpha}\,\mathbf{k} \times \mathbf{n}.$$

By the torque law, we therefore have

$$I\dot{\theta}\dot{\alpha} \simeq c(T_2 - T_1)$$

or $\quad |\dot{\alpha}| \simeq \dfrac{c(T_1 - T_2)}{I|\dot{\theta}|}.$

It is also known that

$$T_1 = k(l_1 - l_0), \quad T_2 = k(l_2 - l_0) \quad \text{and} \quad I = \tfrac{1}{2}MR^2,$$

leading to the conclusion that

$$|\dot{\alpha}| \simeq \frac{2ck(l_1 - l_2)}{MR^2|\dot{\theta}|},$$

as required.

Solutions to the exercises in Section 5

1. Modelling the man by a particle, the system is composed of two rigid bodies (the disc and the man) whose total angular momentum is initially zero. When the man walks, the total angular momentum is

$$I_1\dot{\theta}_1 + I_2\dot{\theta}_2,$$

where I_1, I_2 are the moments of inertia of the disc and man respectively, and $\dot{\theta}_1, \dot{\theta}_2$ are the corresponding angular velocities (about the vertical axis through the centre of the disc in each case). No external torque acts, so conservation of angular momentum applies, giving

$$I_1\dot{\theta}_1 + I_2\dot{\theta}_2 = 0. \qquad (1)$$

This shows that $\dot{\theta}_1$ and $\dot{\theta}_2$ have opposite signs. Since the man walks in an anticlockwise sense (with $\dot{\theta}_2 > 0$), the disc rotates in a clockwise sense (with $\dot{\theta}_1 < 0$). The speed of the man is

$$v_2 = r|\dot{\theta}_2| = r\dot{\theta}_2,$$

while the speed of the portion of the disc immediately beneath the man is

$$v_1 = r|\dot{\theta}_1| = -r\dot{\theta}_1.$$

The speed of the man relative to the disc is

$$v = v_2 + v_1 = r\dot{\theta}_2 - r\dot{\theta}_1,$$

leading to

$$\dot{\theta}_2 = \frac{v}{r} + \dot{\theta}_1.$$

Substituting for $\dot{\theta}_2$ in Equation (1), we obtain

$$I_1\dot{\theta}_1 + I_2\left(\frac{v}{r} + \dot{\theta}_1\right) = 0,$$

giving

$$\dot{\theta}_1 = -\frac{I_2 v/r}{I_1 + I_2}.$$

However, the moments of inertia of the disc and particle are respectively

$$I_1 = \tfrac{1}{2}MR^2 \quad \text{and} \quad I_2 = mr^2,$$

so that the disc has angular velocity

$$\dot{\theta}_1 = -\frac{mvr}{\frac{1}{2}MR^2 + mr^2}.$$

2. We denote by θ the angle between the downward vertical and the line which joins the mid-point of the rod to the centre O of the circular track. The distance between O and the mid-point of the rod is $c = \sqrt{b^2 - a^2}$.

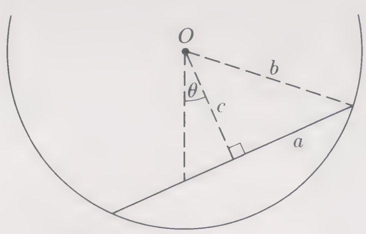


Figure 1

The kinetic energy of the rod is $T = \frac{1}{2}I\dot{\theta}^2$, where I is the moment of inertia of the rod about an axis through O and perpendicular to the plane of the track. Table 1 of Section 2 (with $h = 2a$) and the parallel axes theorem give

$$I = \tfrac{1}{3}ma^2 + mc^2 = \tfrac{1}{3}m(3b^2 - 2a^2),$$

where m is the mass of the rod. The potential energy is given by

$$U = -mgc\cos\theta$$

(choosing $\theta = \frac{1}{2}\pi$ as the datum). Thus the constant total mechanical energy $E = T + U$ is given by

$$E = \tfrac{1}{6}m(3b^2 - 2a^2)\dot{\theta}^2 - mgc\cos\theta.$$

Differentiating with respect to time and then dividing by $\dot{\theta}$, we obtain

$$\tfrac{1}{3}m(3b^2 - 2a^2)\ddot{\theta} + mgc\sin\theta = 0.$$

Using the approximation $\sin\theta \simeq \theta$ for small oscillations leads to the simple harmonic equation

$$\ddot{\theta} + \omega^2\theta = 0,$$

where

$$\omega^2 = \frac{3gc}{3b^2 - 2a^2} = \frac{3g\sqrt{b^2 - a^2}}{3b^2 - 2a^2}.$$

The angular frequency ω of small oscillations is the square root of this expression.

3. **(i)** The diagram below indicates the rod and the forces acting on it, together with the chosen coordinate system. The polar coordinate unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are indicated for use in part (ii).

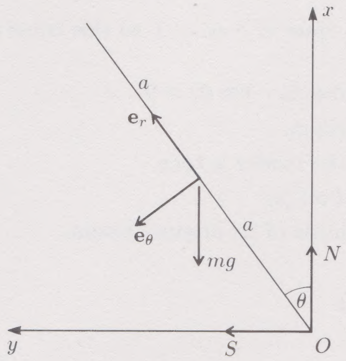


Figure 2

The kinetic energy of the rod is $T = \frac{1}{2}I\dot{\theta}^2$ where, from Table 1 of Section 2 (with $h = 2a$) and the parallel axes theorem, the moment of inertia of the rod about the axis of rotation through O is

$$I = \tfrac{1}{3}ma^2 + ma^2 = \tfrac{4}{3}ma^2.$$

The potential energy is

$$U = mga\cos\theta$$

(taking $\theta = \frac{1}{2}\pi$ as the datum). The constant total mechanical energy $E = T + U$ has value mga when $\theta = 0$ (and $\dot{\theta} = 0$), so that

$$\tfrac{2}{3}ma^2\dot{\theta}^2 + mga\cos\theta = mga.$$

This is equivalent to

$$\dot{\theta}^2 = \frac{3g}{2a}(1 - \cos\theta).$$

Differentiation and division by $2\dot{\theta}$ gives

$$\ddot{\theta} = \frac{3g}{4a}\sin\theta,$$

as required.

(ii) Using the given hint, with $\mathbf{R} = a\mathbf{e}_r$ in place of $\mathbf{r} = r\mathbf{e}_r$, we have

$$\begin{aligned}\ddot{\mathbf{R}} &= -a\dot{\theta}^2\mathbf{e}_r + a\ddot{\theta}\mathbf{e}_\theta \\ &= -a\dot{\theta}^2(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) + a\ddot{\theta}(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) \\ &= a(-\dot{\theta}^2\cos\theta - \ddot{\theta}\sin\theta)\mathbf{i} + a(-\dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta)\mathbf{j}.\end{aligned}$$

Assuming that the frictional force is directed forwards, as shown in the diagram above, the total external force on the rod is

$$\mathbf{F}^{\text{ext}} = mg(-\mathbf{i}) + N\mathbf{i} + S\mathbf{j} = (N - mg)\mathbf{i} + S\mathbf{j}.$$

From the equation $m\ddot{\mathbf{R}} = \mathbf{F}^{\text{ext}}$, we therefore obtain

$$N = mg - ma(\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta),$$
$$S = ma(-\dot{\theta}^2\sin\theta + \ddot{\theta}\cos\theta).$$

Substituting the expressions for $\dot{\theta}^2$ and $\ddot{\theta}$ obtained in part (i) gives

$$N = \tfrac{1}{4}mg(1 - 3\cos\theta)^2,$$
$$S = \tfrac{3}{4}mg\sin\theta(3\cos\theta - 2).$$

However, the second expression is positive only for $3\cos\theta > 2$, or for $\theta < \arccos\frac{2}{3}$. For larger angles, the direction of the frictional force is reversed, giving its magnitude S as the negative of the expression above. In either case, we have

$$S = \tfrac{3}{4}mg\sin\theta|3\cos\theta - 2|,$$

as required.

(iii) Consider the ratio

$$\rho = \frac{S}{N} = \frac{3\sin\theta|3\cos\theta - 2|}{(1 - 3\cos\theta)^2}.$$

The condition for the lower end of the rod to remain static is $\rho \leq \mu$. The ratio ρ is zero at $\theta = 0$ and at $\theta = \arccos\frac{2}{3}$, at which times there is no friction. Between these values of θ the ratio is positive, so it has a positive overall maximum (μ_0, say) somewhere within this interval. (In fact, $\mu_0 \simeq 0.37$, which is the value of ρ at $\arccos\frac{9}{11}$.)

(a) If $\mu < \mu_0$, then ρ will exceed μ for some value of θ between 0 and $\arccos\frac{2}{3}$. In this interval the frictional force acts forwards, as discussed in part (ii), so the rod will slip backwards.

(b) If $\mu \geq \mu_0$, then $\rho \leq \mu$ for all values of θ less than $\arccos\frac{2}{3}$, and no sliding takes place in this interval. However, once $\theta = \arccos\frac{2}{3}$ is passed and the direction of the frictional force is reversed (now acting backwards), the ratio ρ increases without bound as $\theta \to \arccos\frac{1}{3}$. Hence before this angle is reached, the value of ρ will exceed μ, and the rod will slip forwards.

4. As a preliminary, we establish the rolling condition for this situation (which differs from the condition given in the text, since here the rolling takes place on a non-planar surface).

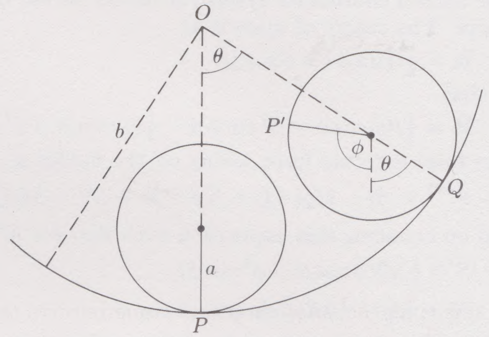


Figure 3

Suppose that θ is the angle between the downward vertical and the line which joins the centre O of the track to the centre of the ball-bearing, as in the diagram above. As the point of contact between ball-bearing and track moves from P (the lowest point of the track) to Q, the point of the ball-bearing which was initially at P moves to P', and the ball-bearing rotates through an angle ϕ. Then, since no slipping occurs, the lengths of the circular arcs PQ and $P'Q$ are equal, that is,

$$b\theta = a(\theta + \phi).$$

It follows by differentiation that

$$a\dot{\phi} = (b - a)\dot{\theta},$$

which is the rolling condition. By the centre of mass decomposition theorem, the kinetic energy of the ball-bearing (of mass m) is

$$\begin{aligned}T &= T_G + \tfrac{1}{2}m|\dot{\mathbf{R}}|^2 \\ &= \tfrac{1}{2}I\dot{\phi}^2 + \tfrac{1}{2}m(b - a)^2\dot{\theta}^2,\end{aligned}$$

where the moment of inertia of the ball-bearing (a solid sphere) about its centre of mass is $I = \frac{2}{5}ma^2$ from Table 1 in Section 2. It follows from the rolling condition that

$$T = \tfrac{1}{5}m(b - a)^2\dot{\theta}^2 + \tfrac{1}{2}m(b - a)^2\dot{\theta}^2 = \tfrac{7}{10}m(b - a)^2\dot{\theta}^2.$$

The potential energy (with datum at $\theta = \frac{1}{2}\pi$) is

$$U = -mg(b - a)\cos\theta.$$

Hence the constant total mechanical energy $E = T + U$ is given by

$$E = \tfrac{7}{10}m(b - a)^2\dot{\theta}^2 - mg(b - a)\cos\theta.$$

Differentiation and simplification then gives the equation

$$\tfrac{7}{5}(b - a)\ddot{\theta} + g\sin\theta = 0.$$

Applying the approximation $\sin\theta \simeq \theta$ for small oscillations produces the simple harmonic equation

$$\ddot{\theta} + \omega^2\theta = 0,$$

where

$$\omega = \sqrt{\frac{5g}{7(b - a)}}$$

is the frequency of the oscillations.

5. (i)

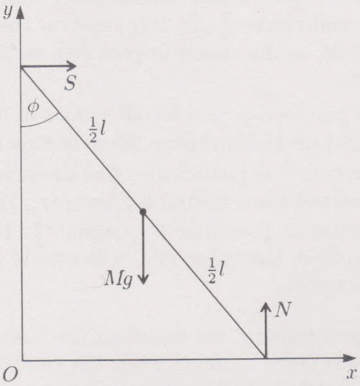


Figure 4

The chosen coordinate system is shown on the diagram above. The centre of mass is at

$$\mathbf{R} = \tfrac{1}{2}l(\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}),$$

so that

$$\ddot{\mathbf{R}} = \tfrac{1}{2}l(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi)\mathbf{i} - \tfrac{1}{2}l(\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi)\mathbf{j}.$$

The total external force acting on the ladder is

$$\mathbf{F}^{\text{ext}} = S\mathbf{i} + Mg(-\mathbf{j}) + N\mathbf{j} = S\mathbf{i} + (N - Mg)\mathbf{j},$$

and on equating this expression with that for $M\ddot{\mathbf{R}}$, we obtain

$$S = \tfrac{1}{2}Ml(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi), \qquad (2)$$

$$N = Mg - \tfrac{1}{2}Ml(\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi). \qquad (3)$$

The moment of inertia of the ladder relative to its centre of mass is $I = \frac{1}{12}Ml^2$ by Table 1 of Section 2, so the angular momentum relative to the centre of mass is

$$\mathbf{L}_G = I\dot{\phi}\mathbf{k} = \tfrac{1}{12}Ml^2\dot{\phi}\,\mathbf{k}.$$

The external torque relative to the centre of mass is

$$\boldsymbol{\Gamma}_G = (\tfrac{1}{2}l\cos\phi\,\mathbf{j}\times S\mathbf{i}) + (\tfrac{1}{2}l\sin\phi\,\mathbf{i}\times N\mathbf{j})$$
$$= \tfrac{1}{2}l(-S\cos\phi + N\sin\phi)\mathbf{k}.$$

From the torque law $\dot{\mathbf{L}}_G = \boldsymbol{\Gamma}_G$, we therefore have

$$\tfrac{1}{6}Ml\ddot{\phi} = -S\cos\phi + N\sin\phi.$$

On substituting for S and N from Equations (2) and (3), we obtain

$$\tfrac{1}{6}Ml\ddot{\phi} = -\tfrac{1}{2}Ml\cos\phi(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi)$$
$$+ Mg\sin\phi - \tfrac{1}{2}Ml\sin\phi(\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi),$$

giving the equation of motion

$$\ddot{\phi} = \frac{3g}{2l}\sin\phi,$$

as required.

(ii) Multiplying the previous equation by $\dot{\phi}$ produces

$$\dot{\phi}\ddot{\phi} = \frac{3g}{2l}\dot{\phi}\sin\phi.$$

Integrating with respect to time gives

$$\int \frac{d}{dt}(\tfrac{1}{2}\dot{\phi}^2)dt = \int \frac{3g}{2l}\sin\phi\,d\phi + C$$

or $\quad \tfrac{1}{2}\dot{\phi}^2 = C - \dfrac{3g}{2l}\cos\phi,$

where C is a constant. Now $\dot{\phi} = 0$ when $\phi = \phi_0$, from which $C = 3g\cos\phi_0/(2l)$. Hence we have

$$\dot{\phi}^2 = \frac{3g}{l}(\cos\phi_0 - \cos\phi),$$

as required.

(iii) The ladder leaves the wall when $S = 0$, which from Equation (2) occurs when

$$\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi = 0.$$

Using the results of parts (i) and (ii), this becomes

$$\frac{3g}{2l}\sin\phi\cos\phi - \frac{3g}{l}(\cos\phi_0 - \cos\phi)\sin\phi = 0.$$

Now $\sin\phi > 0$, since $\phi \geq \phi_0 > 0$, so this equation is equivalent to

$$\tfrac{1}{2}\cos\phi - (\cos\phi_0 - \cos\phi) = 0$$

or $\quad \cos\phi = \tfrac{2}{3}\cos\phi_0.$

The height of the ladder is then

$$l\cos\phi = \tfrac{2}{3}l\cos\phi_0,$$

which is two thirds of its original height.

6. (i)

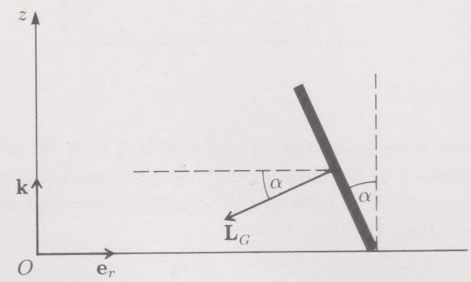


Figure 5

In the diagram above the wheel is moving into the paper, and the total angular momentum relative to the centre of mass is directed along the spin axis as shown, being given by

$$\mathbf{L}_G \simeq I\dot{\phi}\,\mathbf{n},$$

where $I = Ma^2$ is the moment of inertia of the wheel relative to its spin axis (assuming that all the mass of the wheel is concentrated in its rim), and $\dot{\phi}\,\mathbf{n}$ is the angular velocity of the wheel. The unit vector $\mathbf{n}$ is given by

$$\mathbf{n} = -\cos\alpha\,\mathbf{e}_r - \sin\alpha\,\mathbf{k},$$

where

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

is the unit vector from O towards the base of the wheel, as shown in the figure below.

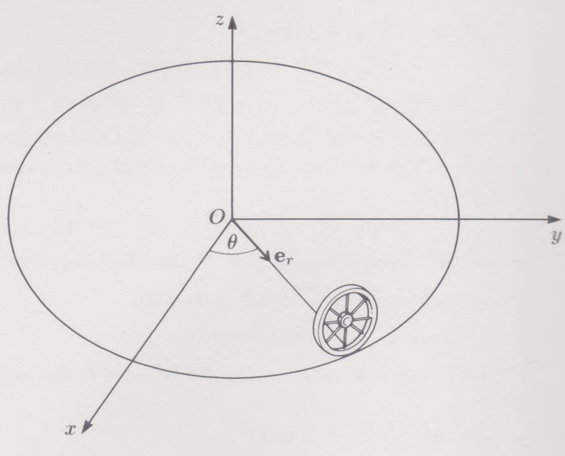


Figure 6

The rolling condition for the wheel is $v = a\dot{\phi}$ so, from above, the total angular momentum relative to the centre of mass is

$$\mathbf{L}_G \simeq Ma^2\frac{v}{a}(-\cos\alpha\,\mathbf{e}_r - \sin\alpha\,\mathbf{k})$$

or $\quad \mathbf{L}_G \simeq Mav\,(-\cos\alpha\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) - \sin\alpha\,\mathbf{k})\,.$

(ii) Following exactly the same argument as in Exercise 9(i) of Section 3, we have

$$\boldsymbol{\Gamma}_G = Ma\cos\alpha\left(\frac{v^2}{R} - g\tan\alpha\right)\mathbf{e}_\theta,$$

where

$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

is a unit vector in the direction of motion of the wheel's centre of mass. Hence we find that

$$\boldsymbol{\Gamma}_G = Ma\cos\alpha\left(\frac{v^2}{R} - g\tan\alpha\right)(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}).$$

(iii) On differentiating the result of part (i), we obtain

$$\dot{\mathbf{L}}_G \simeq -Mav\dot{\theta}\cos\alpha\,(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$$

or $$\dot{\mathbf{L}}_G \simeq -\frac{Mav^2}{R}\cos\alpha\,(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}),$$

since $v = R\dot{\theta}$. The torque law $\dot{\mathbf{L}}_G = \mathbf{\Gamma}_G$ then gives

$$-\frac{Mav^2}{R}\cos\alpha \simeq Ma\cos\alpha\left(\frac{v^2}{R} - g\tan\alpha\right)$$

or $$\tan\alpha \simeq \frac{2v^2}{gR},$$

as required.

(Comparing this with the result $\tan\alpha = v^2/(gR)$ for the skater in Exercise 9 of Section 3, it can be seen that a rolling object which moves in a circle leans inwards more than a sliding one of the same mass and speed.)

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